# Trigonometry Identities



**P1.** If $\cos\theta + \cos^2\theta = 1$, prove that

$$\sin^{12}\theta + 3\sin^{10}\theta + 3\sin^8\theta + \sin^6\theta + 2\sin^4\theta + 2\sin^2\theta - 2 = 1$$

**Solⁿ.** Given $\cos\theta + \cos^2\theta = 1$

$$\Rightarrow \cos\theta = 1 - \cos^2\theta = \sin^2\theta$$

$$\Rightarrow \cos\theta = \sin^2\theta$$

Now, $\sin^{12}\theta + 3\sin^{10}\theta + 3\sin^8\theta + \sin^6\theta + 2\sin^4\theta + 2\sin^2\theta - 2$

$$= (\sin^2\theta)^6 + 3(\sin^2\theta)^5 + 3(\sin^2\theta)^4 + (\sin^2\theta)^3 + 2(\sin^2\theta)^2 + 2\cos\theta - 2$$

$$= \cos^6\theta + 3\cos^5\theta + 3\cos^4\theta + \cos^3\theta + 2\cos^2\theta - 2 + 2\cos\theta$$

$$= \cos^6\theta + 3\cos^5\theta + 3\cos^4\theta + \cos^3\theta - 2(1 - \cos^2\theta) + 2\cos\theta$$

$$= \cos^6\theta + 3\cos^5\theta + 3\cos^4\theta + \cos^3\theta - 2\cancel{\sin^2\theta} + 2\cancel{\cos\theta} \quad \because \cos\theta = \sin^2\theta.$$

$$= (\cos^2\theta + \cos\theta)^3 \quad [\cos^6\theta + 3\cos^4\theta\cos\theta + 3\cos^2\theta\cos^2\theta]$$

$$= (\cos^2\theta + \sin^2\theta)^3 = 1^3 = 1. \quad \text{Hence proved}$$

**P2.** If $\tan^2\theta = 1 - p^2$ prove that $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta = (2 - p^2)^{3/2}$

**Solⁿ.** $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta = \sec\theta + \tan^2\theta \cdot \tan\theta \cdot \text{cosec}\,\theta$

$$= \frac{1}{\cos\theta} + \tan^2\theta \cdot \frac{\cancel{\sin\theta}}{\cos\theta} \cdot \frac{1}{\cancel{\sin\theta}}$$

$$= \frac{1}{\cos\theta}(1 + \tan^2\theta) = \frac{1}{\cos\theta}(1 + 1 - p^2)$$

$$= \frac{1}{\cos\theta}(2 - p^2) = \sec\theta\,(2 - p^2) \quad \text{--- (i)}$$

Now $\sec^2\theta = 1+\tan^2\theta = 1+1-p^2$
$= 2-p^2$

$\Rightarrow \sec\theta = (2-p^2)^{1/2}$ —— (ii)

$\therefore \sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-p^2)(2-p^2)^{1/2}$ Using (i) & (ii)
$= (2-p^2)^{3/2}$

Proved

P3. If $\sin\theta + \sin^3\theta = \cos^2\theta$, prove that
$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4.$

Solⁿ: Given $\sin\theta + \sin^3\theta = \cos^2\theta$.

$\Rightarrow \sin\theta(1+\sin^2\theta) = \cos^2\theta$.

$\Rightarrow \sin^2\theta(1+\sin^2\theta)^2 = \cos^4\theta$ [Squaring on both side]

$\Rightarrow (1-\cos^2\theta)(1+1-\cos^2\theta)^2 = \cos^4\theta$.

$\Rightarrow (1-\cos^2\theta)(2-\cos^2\theta)^2 = \cos^4\theta$

~~$\Rightarrow 2-\cos^2\theta - 2\cos^2\theta + \cos^4\theta = \cos^4\theta$.~~

$\Rightarrow (1-\cos^2\theta)(4+\cos^4\theta-4\cos^2\theta) = \cos^4\theta$.

$\Rightarrow 4+\cancel{\cos^4\theta} - 4\cos^2\theta - 4\cos^2\theta - \cos^6\theta + 4\cos^4\theta = \cancel{\cos^4\theta}$

$\Rightarrow 4 = \cos^6\theta + 8\cos^2\theta - 4\cos^4\theta$.

Proved

Pb. If $\tan^2\alpha = \cos^2\beta - \sin^2\beta$, prove that
$\cos^2\alpha - \sin^2\alpha = \tan^2\beta$

Soln: Given $\tan^2\alpha = \cos^2\beta - \sin^2\beta$

$\Rightarrow \tan^2\alpha = 1 - \sin^2\beta - \sin^2\beta$

$\Rightarrow \tan^2\alpha = 1 - 2\sin^2\beta$.

$\Rightarrow 2\sin^2\beta = 1 - \tan^2\alpha = \dfrac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}$.

$\Rightarrow 2\sin^2\beta = \dfrac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}$ ——— (A)

Again, $\tan^2\alpha = \cos^2\beta - 1 + \cos^2\beta$

$\Rightarrow \tan^2\alpha = 2\cos^2\beta - 1$

$\Rightarrow 2\cos^2\beta = 1 + \tan^2\alpha$

$\Rightarrow 2\cos^2\beta = \dfrac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha} = \dfrac{1}{\cos^2\alpha}$

$\Rightarrow 2\cos^2\beta = \dfrac{1}{\cos^2\alpha}$ ——— (B)

Dividing (A) by (B) we get

$\tan^2\beta = \dfrac{\cos^2\alpha - \sin^2\alpha}{\cancel{\cos^2\alpha}} \times \dfrac{\cancel{\cos^2\alpha}}{1}$

$\Rightarrow \tan^2\beta = \cos^2\alpha - \sin^2\alpha$ Proved

P5. If $\tan^2\alpha = 1 + 2\tan^2\beta$, prove that
$$2\sin^2\alpha = 1 + \sin^2\beta.$$

Sol$^n$. Given, $\tan^2\alpha = 1 + 2\tan^2\beta$

$$\Rightarrow \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\cos^2\beta + 2\sin^2\beta}{\cos^2\beta}$$

$$\Rightarrow \frac{\sin^2\alpha}{1-\sin^2\alpha} = \frac{1+\sin^2\beta}{1-\sin^2\beta}$$

$$\Rightarrow \sin^2\alpha\,(1-\sin^2\beta) = (1-\sin^2\alpha)(1+\sin^2\beta)$$

$$\Rightarrow \sin^2\alpha - \cancel{\sin^2\alpha\sin^2\beta} = 1 + \sin^2\beta - \sin^2\alpha - \cancel{\sin^2\alpha\sin^2\beta}$$

$$\Rightarrow 2\sin^2\alpha = 1 + \sin^2\beta$$ Proved

P6. If $x = c + a\cos\theta$, $y = d + b\sin\theta$ prove that
$$\left(\frac{x-c}{a}\right)^2 + \left(\frac{y-d}{b}\right)^2 = 1$$

Sol$^n$. Given $x = c + a\cos\theta$ & $y = d + b\sin\theta$

$$\Rightarrow \frac{x-c}{a} = \cos\theta \quad \& \quad \frac{y-d}{b} = \sin\theta$$

Now $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \left(\frac{x-c}{a}\right)^2 + \left(\frac{y-d}{b}\right)^2 = 1.$$

Proved.

P7. If $5\sin\theta + 7\cos\theta = 7$, show that
$$7\sin\theta - 5\cos\theta = \pm 5$$

Sol$^n$. Given $(5\sin\theta + 7\cos\theta) = 7$

$\Rightarrow 25\sin^2\theta + 2.5.7\sin\theta.\cos\theta + 49\cos^2\theta = 49$

$\Rightarrow 25 - 25\cos^2\theta + 2.5.7\sin\theta\cos\theta + 49 - 49\sin^2\theta = 49$

$\Rightarrow 74 - (25\cos^2\theta - 2.5.7\sin\theta\cos\theta + 49\sin^2\theta) = 49$

$\Rightarrow 25 = (25\cos^2\theta - 2.5.7\sin\theta\cos\theta + 49\cos^2\theta)$

$\Rightarrow 25 = (7\sin\theta - 5\cos\theta)^2$

$\Rightarrow 7\sin\theta - 5\cos\theta = \pm 5$. Taking sq. root on both side.

P8. If $\operatorname{cosec}\theta + \cot\theta = m$, prove that
$$\frac{m^2-1}{m^2+1} = \cos\theta.$$

Sol$^n$. $m^2 = \operatorname{cosec}^2\theta + \cot^2\theta + 2\operatorname{cosec}\theta\,\cot\theta$

$\therefore m^2+1 = 1 + \operatorname{cosec}^2\theta + \cot^2\theta + 2\operatorname{cosec}\theta\cot\theta$

$= 2\operatorname{cosec}^2\theta + 2\operatorname{cosec}\theta\cot\theta$ $[\because 1+\cot^2\theta = \operatorname{cosec}^2\theta]$

$\Rightarrow m^2+1 = 2\operatorname{cosec}\theta(\operatorname{cosec}\theta + \cot\theta)$ —— (A)

& $m^2-1 = \operatorname{cosec}^2\theta + \cot^2\theta + 2\operatorname{cosec}\theta\cot\theta - 1$

$= 2\cot^2\theta + 2\operatorname{cosec}\theta\cot\theta$

$\Rightarrow m^2-1 = 2\cot\theta(\cot\theta + \operatorname{cosec}\theta)$ $[\because \operatorname{cosec}^2\theta - 1 = \cot^2\theta]$ —— (B)

Dividing (B) by (A) we get

$$\frac{m^2-1}{m^2+1} = \frac{\cancel{2}\cot\theta\,\cancel{(\operatorname{cosec}\theta+\cot\theta)}}{\cancel{2}\operatorname{cosec}\theta\,\cancel{(\operatorname{cosec}\theta+\cot\theta)}}$$

$$= \frac{\cos\theta}{\cancel{\sin\theta}} \times \cancel{\sin\theta} = \cos\theta.$$

$$\Rightarrow \frac{m^2-1}{m^2+1} = \cos\theta. \text{ Proved}$$

P9. If $p = \sec\theta - \operatorname{cosec}\theta$ & $q = \sin\theta - \cos\theta$, then prove that $p(q^2-1) + 2q = 0$.

Soln:

$$q^2 - 1 = \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta - 1$$

$$= \cancel{1} - 2\sin\theta\cos\theta - \cancel{1}$$

$$= -2\sin\theta\cos\theta.$$

$$p = \sec\theta - \operatorname{cosec}\theta = \frac{1}{\cos\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\sin\theta - \cos\theta}{\sin\theta\cos\theta}.$$

$$\therefore p(q^2-1) + 2q = \frac{(\sin\theta - \cos\theta)}{\cancel{\sin\theta\cos\theta}} \times (-2\cancel{\sin\theta\cos\theta}) + 2q$$

$$= 2(\cos\theta - \sin\theta) + 2(\sin\theta - \cos\theta)$$

$$= 2\{\cancel{\cos\theta} - \cancel{\sin\theta} + \cancel{\sin\theta} - \cancel{\cos\theta}\}$$

$$= 2 \times 0 = 0. \text{ Proved}$$

P10. Prove that $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta)$
$= \sec\theta + \operatorname{cosec}\theta$

Solⁿ

$$\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)$$

$$= \sin\theta\left(1+\frac{\sin\theta}{\cos\theta}\right)+\cos\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin\theta\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)+\cos\theta\left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)$$

$$= (\sin\theta+\cos\theta)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= (\sin\theta+\cos\theta)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \frac{\sin\theta+\cos\theta}{\sin\theta\cdot\cos\theta}$$

$$= \frac{1}{\cos\theta}+\frac{1}{\sin\theta} = \sec\theta+\operatorname{cosec}\theta$$

Proved

P.11 Prove that $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta} = 1+\sec\theta\operatorname{cosec}\theta$

Solⁿ

$$\frac{\tan\theta}{1-\cot\theta}+\frac{\cot\theta}{1-\tan\theta}$$

$$= \frac{\tan\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cot\theta}{1-\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\tan\theta}{\frac{\sin\theta-\cos\theta}{\sin\theta}}+\frac{\cot\theta}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{\sin\theta\cdot\tan\theta}{(\sin\theta-\cos\theta)} - \frac{\cos\theta\cdot\cot\theta}{(\sin\theta-\cos\theta)}$$

$$= \frac{1}{(\sin\theta - \cos\theta)} \left[ \sin\theta \cdot \frac{\sin\theta}{\cos\theta} - \cos\theta \cdot \frac{\cos\theta}{\sin\theta} \right]$$

$$= \frac{1}{(\sin\theta - \cos\theta)} \left( \frac{\sin^3\theta - \cos^3\theta}{\cos\theta \cdot \sin\theta} \right)$$

$$\cancel{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}$$

$$= \frac{1}{\cancel{(\sin\theta - \cos\theta)}} \frac{\cancel{(\sin\theta - \cos\theta)}(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)}{\cos\theta \cdot \sin\theta}$$

$$= \frac{\cancel{\sin^2\theta}\,\sin\theta}{\cos\theta \cdot \cancel{\sin\theta}} + \frac{1\,\cancel{\sin\theta\cos\theta}}{\cancel{\sin\theta\cos\theta}} + \frac{\cancel{\cos^2\theta}\,\cos\theta}{\sin\theta\,\cancel{\cos\theta}}$$

$$= \tan\theta + 1 + \cot\theta$$

$$= 1 + \tan\theta + \cot\theta.$$

P12. Prove that $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$$= 7 + \tan^2\theta + \cot^2\theta$$

Sol: $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$$= \sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta \cdot \operatorname{cosec}\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta \cdot \sec\theta$$

$$= \sin^2\theta + \cos^2\theta + 1 + \cot^2\theta + 2 + 1 + \tan^2\theta + 2$$

$$= 1 + 6 + \cot^2\theta + \tan^2\theta$$

$$= 7 + \cot^2\theta + \tan^2\theta.$$

P13 Prove that

$$\left(\sin\theta + \frac{\cos\theta}{\sec\theta}\right)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = (1 + \sec\theta\operatorname{cosec}\theta)^2$$

Solⁿ: $\left(\sin\theta + \frac{\cancel{\cos\theta}}{\sec\theta}\right)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$$= \sin^2\theta + \frac{\cancel{\cos^2\theta}}{\sec^2\theta} + 2\sin\theta\sec\theta + \cos^2\theta + \operatorname{cosec}^2\theta + 2\cos\theta\cdot\operatorname{cosec}\theta$$

$$= (\sin^2\theta + \cos^2\theta) + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$$

$$= 1 + \frac{\sin^2\theta + 2\cos\theta\sin^3\theta + \cos^2\theta + 2\cos^3\theta\sin\theta}{\cos^2\theta\sin^2\theta}$$

$$= 1 + \frac{1 + 2\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)}{\cos^2\theta\sin^2\theta}$$

$$= 1 + \frac{1}{\cos^2\theta\sin^2\theta} + \frac{\cancel{2\sin\theta\cos\theta}}{\cancel{\cos^2\theta\sin^2\theta}}$$
$$\cos\theta\sin\theta$$

$$= 1 + \sec^2\theta\operatorname{cosec}^2\theta + 2\sec\theta\operatorname{cosec}\theta$$

$$= (1 + \sec\theta\operatorname{cosec}\theta)^2 \quad \text{Proved}$$

P14 If $\frac{\cos\alpha}{\cos\beta} = m$ & $\frac{\cos\alpha}{\sin\beta} = n$, then show that $(m^2+n^2)\cos^2\beta = n^2$

Solⁿ: $(m^2+n^2)\cos^2\beta = \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$

$$= \cos^2\alpha\left(\frac{\sin^2\beta + \cos^2\beta}{\cancel{\cos^2\beta}\sin^2\beta}\right)\cancel{\cos^2\beta}$$

$$= \frac{\cos^2\alpha}{\sin^2\beta} = n^2 \quad \text{Proved}$$

P15 Prove that

$$\frac{1}{\operatorname{cosec}\theta+\cot\theta}-\frac{1}{\sin\theta}=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta-\cot\theta}$$

Sol$^n$: L.H.S will be equal to R.H.S iff

$$\frac{1}{\operatorname{cosec}\theta+\cot\theta}+\frac{1}{\operatorname{cosec}\theta-\cot\theta}=\frac{2}{\sin\theta}.$$

So we prove the above identity

$$\frac{1}{\operatorname{cosec}\theta+\cot\theta}+\frac{1}{\operatorname{cosec}\theta-\cot\theta}$$

$$=\frac{\operatorname{cosec}\theta-\cancel{\cot\theta}+\operatorname{cosec}\theta+\cancel{\cot\theta}}{(\operatorname{cosec}\theta+\cot\theta)(\operatorname{cosec}\theta-\cot\theta)}$$

$$=\frac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta-\cot^2\theta}=\frac{2\operatorname{cosec}\theta}{1}$$

$$=\frac{2}{\sin\theta}$$ Proved.

P16 If $\sin\theta+\sin^2\theta=1$, prove that

$$\cos^2\theta+\cos^4\theta=1.$$

Sol$^n$: Given $\sin\theta+\sin^2\theta=1$

$\Rightarrow \sin\theta=1-\sin^2\theta=\cos^2\theta.$

$\Rightarrow \sin\theta=\cos^2\theta.$

Now, $\cos^2\theta+\cos^4\theta=\cos^2\theta+(\cos^2\theta)^2$

$=\cos^2\theta+(\sin\theta)^2$

$=\cos^2\theta+\sin^2\theta=1.$ Proved.