Problems for Indian Statistical Institute and CMI

1) Let $a_1 > a_2 > ..... > a_n$ be positive real numbers. Compute

$$\underset{n\to\infty}{lt}\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}$$

Soln:

$$\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}<\left(a_1^{\,n}+a_1^{\,n}+.....+a_1^{\,n}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}<\left(ra_1^{\,n}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}<\left(r\right)^{\frac{1}{n}}a_1$$

Again

$$\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}=\left(a_1^{\,n}\left(1+\frac{a_2^{\,n}}{a_1^{\,n}}+.....+\frac{a_r^{\,n}}{a_1^{\,n}}\right)^{\frac{1}{n}}\right)$$

$$\Rightarrow\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}=a_1\left(1+\frac{a_2^{\,n}}{a_1^{\,n}}+.....+\frac{a_r^{\,n}}{a_1^{\,n}}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}>a_1$$

Since

$$1+\frac{a_2^{\,n}}{a_1^{\,n}}+.....+\frac{a_r^{\,n}}{a_1^{\,n}}>1$$

Therefore

$$a_1<\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}<\left(r\right)^{\frac{1}{n}}a_1$$

By sandwich Theorem

$$\underset{n\to\infty}{lt}\left(a_1^{\,n}+a_2^{\,n}+.....+a_r^{\,n}\right)^{\frac{1}{n}}=a_1$$

Note: $\underset{n\to\infty}{lt}\left(r\right)^{\frac{1}{n}}=1$

2. Let ω denote a cube root of unity which is not equal to 1. Find the number of distinct elements in the set $\left\{\left(1+\omega+\omega^2+.....+\omega^n\right)^m : m,n=1,2,3,.....\right\}$

Soln: Case 1: $n=3k, k\in N$

$$\left(1+\omega+\omega^2+.....+\omega^n\right)^m=\left(\left(\overline{1+\omega+\omega^2}\right)+\left(\overline{1+\omega+\omega^2}\right)+.....+\left(\overline{1+\omega+\omega^2}\right)(k\ times)\right)^m$$
$$=0$$
Case 2: $n=3k+1, k\in N$
$$\left(1+\omega+\omega^2+.....+\omega^n\right)^m=\left(\left(\overline{1+\omega+\omega^2}\right)+\left(\overline{1+\omega+\omega^2}\right)+.....+\left(\overline{1+\omega+\omega^2}\right)(k\ times)\quad +1\right)^m$$
$$=1$$
Case 3: $n=3k+2, k\in N$
$$\left(1+\omega+\omega^2+.....+\omega^n\right)^m=\left(\left(\overline{1+\omega+\omega^2}\right)+\left(\overline{1+\omega+\omega^2}\right)+.....+\left(\overline{1+\omega+\omega^2}\right)(k\ times)\quad +1+\omega\right)^m$$
$$=\left(1+\omega\right)^m$$
$$=\left(-\omega^2\right)^m$$
$$=\omega,\omega^2,1,-1,-\omega,-\omega^2,\ for\ m\in N$$
number of distinct elements in the set is 7

3) The number of perfect cubes among the first 4000 positive integers is?

Soln: Clearly
$15^3=3375$
&
$16^3>4000$
therefore The number of perfect cubes is 15. need more analytical proof!
Consider the function
$$f:\{1,2,.....,15\}\rightarrow\{\ Perfect\ cubes<4000\quad\}$$
$$where\qquad f(n)=n^3$$
Note that f is a bijection

4) The number of integers between 1 and 1000 (both inclusive) which are neither perfect squares nor perfect cubes is?

Soln: We will approach the problem by finding the complement!
Number of perfect squares between 1 and 1000 are 31 (why!)
Number of perfect cubes between 1 and 1000 are 10 (why!)
Adding we get 41 and it seem the answer to be 1000-41=959 which is absolutely wrong till we observe this!
$1=1^2=1^3$
$64=4^3=8^2$
$729=9^3=27^2$
so the correct answer is 959+3=962 (Remember if you have 10 rupee and if you count it twice it will not become 20)!!!!!!

5) The product of the first 100 positive integers ends with?

Soln: Highest power of 2 which divides 100!=

$$\left[\frac{100}{2}\right]+\left[\frac{100}{2^2}\right]+\left[\frac{100}{2^3}\right]+\left[\frac{100}{2^4}\right]+\left[\frac{100}{2^5}\right]+\left[\frac{100}{2^6}\right]$$

$$=50+25+12+6+3+1$$

$$=97$$

Highest power of 2 which divides 100!=

$$\left[\frac{100}{5}\right]+\left[\frac{100}{5^2}\right]$$

$$=20+4$$

$$=24$$

Therefore the products ends with 24 zeroes!

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