(1) Find the value of $\log_2 32$, $\log_3 27$, $\log_2\left(\frac{1}{2}\right)$, $\log_{10}(0.1)$ and $\log_7 343$

(i) $\log_2 32 = \log_2 2^5 = 5\log_2 2 = 5\times1 = 5 \quad (\because \log_2 2 = 1)$

(ii) $\log_3 27 = \log_3 3^3 = 3\log_3 3 = 3\times1 = 3$

(iii) $\log_2\left(\frac{1}{2}\right) = \log_2 1 - \log_2 2 = 0-1 = -1$

(iv) $\log_{10}(0.1) = \log_{10}\left(\frac{1}{10}\right) = \log_{10} 1 - \log_{10} 10 = 0-1 = -1$

(v) $\log_7 343 = \log_7 7^3 = 3\log_7 7 = 3$

(2) If $3^x = 100$, express x in terms of logarithm.

Since, $3^x = 100$

$\Rightarrow x = \log_3 100$ (by definition)

(3) If $\log_x\left(\frac{1}{2}\right) = \frac{1}{3}$, find x.

Since, $\log_x\left(\frac{1}{2}\right) = \frac{1}{3}$

$\Rightarrow \frac{1}{2} = x^{1/3}$ (by definition)

$\Rightarrow x = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(4) Find the value of $\log_{\sqrt{2}} 16$.

$\log_{\sqrt{2}} 16 = \log_{\sqrt{2}} 2^4 = \log_{\sqrt{2}} (\sqrt{2})^8 = 8\log_{\sqrt{2}} \sqrt{2}$

$= 8$

(5) If $\log_2 3 = a$, find $\log_8 27$ in terms of a.

$\log_8 27 = \log_8 3^3 = 3\log_8 3 = 3\log_{2^3} 3$

$= \frac{3}{3}\log_2 3 = \log_2 3 = a$

$\therefore \log_8 27 = a.$

(6) Prove that, $\log(1+3+5+7) = 4\log 2$

$\log(1+3+5+7) = \log 16 = \log 2^4 = 4\log 2$

Proved

(7) Find the value of $\log_9 27 - \log_{27} 9$

$\log_9 27 - \log_{27} 9 = \log_{3^2} 3^3 - \log_{3^3} 3^2$

$= \frac{3}{2}\log_3 3 - \frac{2}{3}\log_3 3$

$= \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6}$

(8) Find b if $\log_{\sqrt{8}} b = 3\frac{1}{3}$

$\log_{\sqrt{8}} b = \frac{10}{3}$

$\Rightarrow (\sqrt{8})^{\frac{10}{3}} = b$ (by definition)

$\Rightarrow 8^{5/3} = b$

$\Rightarrow 2^5 = b$

$\therefore b = 32$

(9) Find the value of $16^{\log_4 5}$

$$16^{\log_4 5} = 4^{2\log_4 5} = 4^{\log_4 5^2}$$

$$= 4^{\log_4 25} = 25.$$

(Since $a^{\log_a x} = x$)

To prove this, let $\log_a x = p$

$\Rightarrow x = a^p$, but $p = \log_a x$

$\therefore a^{\log_a x} = x.$

(10) If $a = \log_{10} 2$, $b = \log_{10} 3$, express the value of $\log_{10} 15$ in terms of a and b.

$$\log_{10} 15 = \log_{10}(3 \times 5) = \log_{10} 3 + \log_{10} 5$$

$$= b + \log_{10}\left(\frac{10}{2}\right)$$

$$= b + \log_{10} 10 - \log_{10} 2$$

$$= b + 1 - a$$

$$= 1 + b - a$$

(11) If $\log_{10} y + 2\log_{10} x = 2$, the show that $y = \frac{100}{x^2}$

Given, $\log_{10} y + 2\log_{10} x = 2$

$\Rightarrow \log_{10} y + \log_{10} x^2 = 2$

$\Rightarrow \log_{10} yx^2 = 2$

$\Rightarrow yx^2 = 10^2$

$\Rightarrow x^2 y = 100$

$\Rightarrow y = 100/x^2$ Proved

(12) Show that $\log_2(\log_3 81) = 2$

$$\log_2(\log_3 81) = \log_2(\log_3 3^4)$$
$$= \log_2(4\log_3 3)$$
$$= \log_2 4 = \log_2 2^2$$
$$= 2\log_2 2 = 2\times 1 = 2$$ Proved

(13) Show that, $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

$$\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = \log_{ab} a + \log_{ab} b \quad \left(\because \log_b a = \frac{1}{\log_a b}\right)$$
$$= \log_{ab}(ab) = 1.$$

(To see that $\log_b a = \frac{1}{\log_a b}$, let

$$x = \log_b a$$
$$\Rightarrow a = b^x.$$
$$\Rightarrow b = a^{1/x}$$
$$\Rightarrow a^{1/x} = b$$
$$\Rightarrow \frac{1}{x} = \log_a b$$
$$\Rightarrow \frac{1}{\log_b a} = \log_a b$$).

(14) Find the value of x, if

$$\log(x+3) = \log x + \log 3$$

Since, $\log(x+3) = \log x + \log 3$

$\Rightarrow \log(x+3) = \log(3x)$

$\Rightarrow 3+x = 3x$

$\Rightarrow 2x = 3$

$\Rightarrow x = 3/2$

(15) Find the value of $\log\left(\frac{b^n}{c^n}\right) + \log\left(\frac{c^n}{a^n}\right) + \log\left(\frac{a^n}{b^n}\right)$

Since $\log(abc) = \log a + \log b + \log c$

$\therefore \log\left(\frac{b^n}{c^n}\right) + \log\left(\frac{c^n}{a^n}\right) + \log\left(\frac{a^n}{b^n}\right)$

$= \log\left(\frac{\cancel{b^n}}{\cancel{c^n}} \times \frac{\cancel{c^n}}{\cancel{a^n}} \times \frac{\cancel{a^n}}{\cancel{b^n}}\right)$

$= \log 1 = 0.$

(16) If $\frac{\log x}{2} = \frac{\log y}{3}$, then show that

$y^4 = x^6$

Since $\frac{\log x}{2} = \frac{\log y}{3} \Rightarrow 3 \log x = 2 \log y$

$\Rightarrow 6 \log x = 4 \log y$

$\Rightarrow \log x^6 = \log y^4$

$\Rightarrow x^6 = y^4$ (Proved)

(17) Find the value in simplest form of

$$\frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}$$

$$= \log_{60} 3 + \log_{60} 4 + \log_{60} 5$$

$$= \log_{60}(3 \times 4 \times 5) = \log_{60} 60 = 1.$$

(18) If $\log\left(\frac{a}{b}\right) + \log\left(\frac{b}{a}\right) = \log(a+b)$, then show that $a+b=1$.

Given, $\log\left(\frac{a}{b}\right) + \log\left(\frac{b}{a}\right) = \log(a+b)$

$$\Rightarrow \log\left(\frac{a}{b} \times \frac{b}{a}\right) = \log(a+b)$$

$$\Rightarrow \log 1 = \log(a+b)$$

$$\Rightarrow a+b = 1.$$

(19) Show that $\log_{\frac{1}{2}} \frac{1}{3} > \log_{\frac{1}{3}} \frac{1}{2}$

Let $\log_{\frac{1}{2}}\left(\frac{1}{3}\right) = x$ and $\log_{\frac{1}{3}}\left(\frac{1}{2}\right) = y$

$\Rightarrow \left(\frac{1}{2}\right)^x = \frac{1}{3}$

$\Rightarrow 2^x = 3$

$\Rightarrow x > 1$

$\Rightarrow \left(\frac{1}{3}\right)^y = \frac{1}{2}$

$\Rightarrow 3^y = 2$

$\Rightarrow y < 1.$

$\therefore x > y$ Proved

(20) If $\log_{12} 27 = a$, then show that

$$\log_6 16 = \frac{4(3-a)}{3+a}.$$

Given, $\log_{12} 27 = a$

$$\Rightarrow \log_{12} 3^3 = a$$

$$\Rightarrow 3\log_{12} 3 = a$$

$$\Rightarrow \frac{3}{\log_3 12} = a$$

$$\Rightarrow \frac{3}{a} = \log_3(12) = \log_3 4 + \log_3 3$$

$$\Rightarrow \frac{3}{a} - 1 = \log_3 4 = \log_3 2^2$$

$$\Rightarrow 2\log_3 2 = \frac{3-a}{a}$$

$$\Rightarrow \log_3 2 = \frac{3-a}{2a}$$

Now, $\log_6 16 = 4\log_6 2 = \dfrac{4}{\log_2 6} = \dfrac{4}{\log_2 2 + \log_2 3}$

$$= \frac{4}{1 + \frac{2a}{3-a}} = \frac{4}{\frac{3-a+2a}{3-a}}$$

$$= \frac{4(3-a)}{3+a}$$

(21) If $\log\left(\frac{x+y}{5}\right) = \frac{1}{2}(\log x + \log y)$, then show that $\frac{x}{y} + \frac{y}{x} = 23$.

Given, $\log\left(\frac{x+y}{5}\right) = \frac{1}{2}(\log x + \log y)$

$\Rightarrow 2\log\left(\frac{x+y}{5}\right) = \log(xy)$

$\Rightarrow \log\left(\frac{x+y}{5}\right)^2 = \log(xy)$

$\Rightarrow \left(\frac{x+y}{5}\right)^2 = \cancel{\log}(xy)$

$\Rightarrow (x+y)^2 = 25xy$

$\Rightarrow x^2 + 2xy + y^2 = 25xy$

$\Rightarrow x^2 + y^2 = 23xy$

$\Rightarrow \frac{x^2}{xy} + \frac{y^2}{xy} = 23$

$\Rightarrow \frac{x}{y} + \frac{y}{x} = 23.$

(22) If $\log_a x = m$, $\log_b x = n$, then show that

$\log_{ab} x = \frac{mn}{m+n}$

$\log_{ab} x = \frac{1}{\log_x ab} = \frac{1}{\log_x a + \log_x b}$

$= \frac{1}{\frac{1}{m} + \frac{1}{n}} = \frac{1}{\frac{m+n}{mn}} = \frac{mn}{m+n}$

Proved

(23) If $a^4+b^4=14a^2b^2$, then show that
$\log(a^2+b^2)=\log a+\log b+2\log 2$

Given, $a^4+b^4=14a^2b^2$

$\Rightarrow (a^2)^2+(b^2)=14a^2b^2$

$\Rightarrow (a^2+b^2)^2-2a^2b^2=14a^2b^2$

$\Rightarrow (a^2+b^2)^2=16a^2b^2$

Take log on both side

$\therefore \log(a^2+b^2)^2=\log(16a^2b^2)$

$\Rightarrow 2\log(a^2+b^2)=\log a^2+\log b^2+\log 2^4$

$\Rightarrow 2\log(a^2+b^2)=2\log a+2\log b+4\log 2$

$\Rightarrow \log(a^2+b^2)=\log a+\log b+2\log 2$

(24) Slove for x: $\log_4 x-\log_4(x-1)=\frac{1}{2}$

$\Rightarrow \log_4 \frac{x}{x-1}=\frac{1}{2}$

$\Rightarrow 4^{\frac{1}{2}}=\frac{x}{x-1}$

$\Rightarrow 2=\frac{x}{x-1}$

$\Rightarrow 2x-2=x$

$\Rightarrow x=2$

(25) If $\log_a bc = x$, $\log_b ca = y$ and $\log_c ab = z$, then find the value of $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$$

$$= \frac{1}{\log_a bc + 1} + \frac{1}{\log_b ca + 1} + \frac{1}{\log_c ab + 1}$$

$$= \frac{1}{\log_a bc + \log_a a} + \frac{1}{\log_b ca + \log_b b} + \frac{1}{\log_c ab + \log_c c}$$

$$= \frac{1}{\log_a (bca)} + \frac{1}{\log_b (cab)} + \frac{1}{\log_c (abc)}$$

$$= \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc}(abc) = 1.$$

(26) Solve for x: $\log_8 x + \log_4 x + \log_2 x = 11$

$$\Rightarrow \frac{1}{\log_x 8} + \frac{1}{\log_x 4} + \frac{1}{\log_x 2} = 11$$

$$\Rightarrow \frac{1}{3\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{\log_x 2} = 11$$

$$\Rightarrow \left(\frac{1}{3} + \frac{1}{2} + 1\right) \times \frac{1}{\log_x 2} = 11$$

$$\Rightarrow \frac{\cancel{11}}{6} \times \log_2 x = \cancel{11}\,1 \Rightarrow \log_2 x = 6$$

$$\Rightarrow x = 2^6 = 64 \text{ Ans}$$

(27) Show that

$$\frac{1}{4} < \log_{10} 2 < \frac{1}{3}$$

Let, $\log_{10} 2 = x$

$\Rightarrow 10^x = 2$

$\Rightarrow (10^x)^{12} = 2^{12}$

$\Rightarrow 10^{12x} = 4096$

Now, $1000 < 4096 < 10000$

$\Rightarrow 10^3 < 4096 < 10^4$

$\Rightarrow 10^3 < 10^{12x} < 10^4$

$\Rightarrow \log_{10} 10^3 < \log_{10} 10^{12x} < \log_{10} 10^4$

$\Rightarrow 3 < 12x < 4$

$\Rightarrow \frac{3}{12} < x < \frac{4}{12}$

$\Rightarrow \frac{1}{4} < \log_{10} 2 < \frac{1}{3}$

(28) If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, then show that

(i) $x^{b+c} \times y^{c+a} \times z^{a+b} = 1$.

(ii) $x^{b^2+bc+c^2} \times y^{c^2+ca+a^2} \times z^{a^2+ab+b^2} = 1$

Let $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b} = k \; (k \neq 0)$

$\Rightarrow \log x = k(b-c)$, $\log y = k(c-a)$, $\log z = k(a-b)$

Now, $\log\left(x^{b+c} \times y^{c+a} \times z^{a+b}\right)$

$= (b+c)\log x + (c+a)\log y + (a+b)\log z$

$= (b+c)k(b-c) + (c+a)k(c-a) + (a+b)k(a-b)$

$= k\left(b^2-c^2+c^2-a^2+a^2-b^2\right)$

$= k \times 0 = 0$.

$\therefore \log\left(x^{b+c} \times y^{c+a} \times z^{a+b}\right) = 0$

$\Rightarrow x^{b+c} \times y^{c+a} \times z^{a+b} = 1$

Similarly proceed for the part (ii).

(29) If $\log_e 2 \cdot \log_x 25 = \log_{10} 16 \cdot \log_e 10$, then find the value of x.

Given, $\log_e 2 \cdot \log_x 25 = \log_e 10 \cdot \log_{10} 16$

$\Rightarrow \log_e 2 \cdot \log_x 25 = \log_e 16$

$\Rightarrow \dfrac{\log_e 2}{\log_e 16} = \dfrac{1}{\log_x 25}$

$\Rightarrow \log_e 2 \times \log_{16} e = \log_{25} x$

$\Rightarrow \log_{16} 2 = \log_{25} x$

$\Rightarrow \log_2 16 = \log_x 25$

$\Rightarrow 4 \log_2 2 = \log_x 25$

$\Rightarrow 4 = \log_x 25$

$\Rightarrow x^4 = 25$

$\Rightarrow x = (25)^{1/4} = (5^2)^{\frac{1}{4}} = 5^{1/2} = \sqrt{5}$.