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Derivatives of Parametric Equations

At times x & y are given as functions of a variable t.

Let $x = \varphi(t)$ & $y = \psi(t)$

Then $\frac{dx}{dt} = \varphi'(t)$ & $\frac{dy}{dt} = \psi'(t)$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\varphi'(t)},$$

$\varphi'(t) \neq 0$.

1. $x = a\cos^3\theta$, $y = b\sin^3\theta$

$$\therefore \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

& $\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\frac{b}{a}\tan\theta \quad \text{Ans}$$

2. $x = a\cos\frac{\theta}{2}$, $y = b\sin\frac{\theta}{2}$

$\therefore \frac{dx}{d\theta} = -\frac{1}{2}a\sin\frac{\theta}{2}$ & $\frac{dy}{d\theta} = \frac{1}{2}b\cos\theta/2$

So, $\frac{dy}{dx} = \dfrac{\frac{1}{2}b\cos\theta/2}{-\frac{1}{2}a\sin\theta/2}$

$= -\frac{b}{a}\cot\theta/2$ Ans

3. $x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$

$\therefore \frac{dx}{dt} = 2a(1 + \cos 2t)$

& $\frac{dy}{dt} = 2a\sin 2t$

So, $\frac{dy}{dx} = \dfrac{\cancel{2}a\sin 2t}{\cancel{2}a(1+\cos 2t)} = \dfrac{\cancel{2}\sin t\cos t}{\cancel{2}\cos^2 t}$

$= \tan t$ Ans

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4. $x = \dfrac{3at}{1+t^3}$, $y = \dfrac{3at^2}{1+t^3}$

$$\therefore \frac{dx}{dt} = 3a \frac{d}{dt}\left(\frac{t}{1+t^3}\right)$$

$$= 3a \frac{(1+t^3).1 - t.3t^2}{(1+t^3)^2}$$

$$= 3a \frac{1-2t^3}{(1+t^3)^2}$$

$$\& \frac{dy}{dt} = 3a . \frac{(1+t^3).2t - t^2.3t^2}{(1+t^3)^2}$$

$$= 3a \frac{2t - t^4}{(1+t^3)^2} = \frac{3at(2-t^3)}{(1+t^3)^2}$$

$$\text{So, } \frac{dy}{dx} = \frac{\cancel{3a}t(2-t^3)}{\cancel{(1+t^3)^2}} \times \frac{\cancel{(1+t^3)^2}}{\cancel{3a}(1-2t^3)}$$

$$= \frac{t(2-t^3)}{1-2t^3} \quad \underline{\text{Ans}}$$

(5) $x = a\left(\frac{1-t^2}{1+t^2}\right)$, $y = b\,\frac{2t}{1+t^2}$

Put $t = \tan\theta$

$\Rightarrow x = a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ & $y = b\,\frac{2\tan\theta}{1+\tan^2\theta}$

$\Rightarrow x = a\cos 2\theta$ & $y = b\sin 2\theta$

$\therefore \frac{dx}{d\theta} = -2a\sin 2\theta$

& $\frac{dy}{d\theta} = 2b\cos 2\theta$

So, $\frac{dy}{dx} = -\frac{2b\cos 2\theta}{2a\sin 2\theta}$

$= -\frac{b}{a\tan 2\theta} = -\frac{b}{a}\,\frac{1-\tan^2\theta}{2\tan\theta}$

$= -\frac{b}{a}\left(\frac{1-t^2}{2t}\right)$ Ans

$(\because \tan\theta = t)$

(6) $x = a(2\cos t + \cos 2t)$, $y = a(2\sin t - \sin 2t)$

$$\therefore \frac{dx}{dt} = a(-2\sin t - 2\sin 2t)$$
$$= -2a\sin t(1+2\cos t)$$

$$\& \frac{dy}{dt} = a(2\cos t - 2\cos 2t)$$
$$= 2a(\cos t - \cos 2t)$$

$$\therefore \frac{dy}{dx} = -\frac{\cancel{2a}(\cos t - \cos 2t)}{\cancel{2a}\sin t(1+2\cos t)}$$

$$= \frac{\cos 2t - \cos t}{\sin t(1+2\cos t)}$$

$$= \frac{2\cos^2 t - \cos t - 1}{\sin t(1+2\cos t)}$$

$$= \frac{(\cos t - 1)\cancel{(2\cos t + 1)}}{\sin t\cancel{(1+2\cos t)}}$$

$$= \frac{-2\sin^2 t/2}{2\sin t/2 \cos t/2} = -\tan t/2$$

Ans

(7) $x = a\sec^2\theta, \; y = a\tan^3\theta$

$$\therefore \frac{dx}{d\theta} = 2a\sec\theta \cdot \sec\theta \cdot \tan\theta$$

$$\& \; \frac{dy}{d\theta} = 3a\tan^2\theta \cdot \sec^2\theta.$$

$$\text{So, } \frac{dy}{dx} = \frac{3\cancel{a}\tan^2\theta \cdot \cancel{\sec^2\theta}}{2\cancel{a}\cancel{\sec^2\theta} \cdot \tan\theta}$$

$$= \frac{3}{2}\tan\theta$$

(8) $x = \ln\left(\tan\frac{t}{2}\right), \; y = \sin t$

$$\therefore \frac{dx}{dt} = \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{2}\,\frac{\sec^2 t/2}{\tan t/2} = \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{1}{\sin t}$$

& $\frac{dy}{dt} = \cos t$

So, $\frac{dy}{dx} = \frac{\cos t}{\frac{1}{\sin t}} = \sin t \cos t$

(9) $x = t \ln t$, $y = \frac{\ln t}{t}$ at $t=1$

$\therefore \frac{dx}{dt} = \frac{t}{t} + \ln t$

$= 1 + \ln t$

& $\frac{dy}{dt} = \frac{t \cdot \frac{1}{t} - \ln t \cdot 1}{t^2}$

$= \frac{1 - \ln t}{t^2}$

So, $\frac{dy}{dx} = \frac{1 - \ln t}{t^2 (1 + \ln t)}$

$\therefore \left.\frac{dy}{dx}\right|_{t=1} = \frac{1 - \ln 1}{1 \cdot (1 + \ln 1)}$

$= \frac{1}{1} = 1$ Ans

(10) $x = \sqrt{\sin 2t}$, $y = \sqrt{\cos 2t}$ at $t = \frac{\pi}{6}$

$\Rightarrow x^2 = \sin 2t$ & $y^2 = \cos 2t$

Now, $\sin^2 2t + \cos^2 2t = 1$

$\therefore x^4 + y^4 = 1$

diff^n both side w.r.t 'x'

we get

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^3$$

$$= -\left(\frac{\sin 2t}{\cos 2t}\right)^{3/2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\left(\frac{\sin \pi/3}{\cos \pi/3}\right)^{3/2}$$

$$= -\left(\tan \pi/3\right)^{3/2}$$

$$= -\left(\sqrt{3}\right)^{3/2}$$ Ans

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(11) $x = 2a\sin^2 t\cos 2t$, $y = 2a\sin^2 t\sin 2t$ when $t = \frac{\pi}{12}$

$$\therefore \frac{dx}{dt} = 2a\frac{d}{dt}(\sin^2 t\cos 2t)$$

$$= 2a\left(\sin^2 t(-2\sin 2t) + \cos 2t \cdot 2\sin t\cos t\right)$$

$$= 2a\left(-2\sin 2t\sin^2 t + \cos 2t\sin 2t\right)$$

$$= 2a\sin 2t\left(-2\sin^2 t + \cos 2t\right)$$

$$= 2a\sin 2t\left(2\cos 2t - 1 + \cos 2t\right)$$

$$= 2a\sin 2t\left(2\cos 2t - 1\right)$$

$$\& \frac{dy}{dt} = 2a\left(2\sin^2 t\cos 2t + \sin 2t \cdot 2\sin t\cos t\right)$$

$$= 2a\left(2\sin^2 t\cos 2t + \sin^2 2t\right)$$

$$= 2a\left((1-\cos 2t)\cdot\cos 2t + \sin^2 2t\right)$$

$$= 2a\left(\cos 2t - \cos^2 2t + \sin^2 2t\right)$$

$$= 2a\left(\cos 2t - \cos 4t\right)$$

$$\therefore \frac{dy}{dx} = \frac{\cancel{2a}(\cos 2t - \cos 4t)}{\cancel{2a}\sin 2t(2\cos 2t - 1)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{12}} = \frac{\cos \pi/6 - \cos \pi/3}{\sin\frac{\pi}{6}\left(2\cos\frac{\pi}{6} - 1\right)} = \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{1}{2}(\sqrt{3} - 1)}$$

$$= 1. \text{ Ans}$$

(12) $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$

$$\therefore \frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\& \frac{dy}{d\theta} = -a\sin\theta$$

$$\text{So, } \frac{dy}{dx} = \frac{-\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} = -\frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^2\theta/2}$$

$$= -\tan\theta/2 \quad \text{Ans}$$

(13) $x = \dfrac{\sin^3\theta}{\sqrt{\cos 2\theta}}$, $y = \dfrac{\cos^3\theta}{\sqrt{\cos 2\theta}}$

$$\therefore \frac{dx}{d\theta} = \frac{\sqrt{\cos 2\theta}\cdot 3\sin^2\theta\cdot\cos\theta - \sin^3\theta \times \frac{1}{2\sqrt{\cos 2\theta}} \times (-2\sin 2\theta)}{\cos 2\theta}$$

$$= \frac{3\sqrt{\cos 2\theta}\sin^2\theta\cos\theta + \dfrac{\sin^3\theta\sin 2\theta}{\sqrt{\cos 2\theta}}}{\cos 2\theta}$$

$$= \frac{3\cos 2\theta\sin^2\theta\cos\theta + \sin^3\theta\sin 2\theta}{\sqrt{\cos 2\theta}\cos 2\theta}$$

$$\& \frac{dy}{d\theta} = \frac{\sqrt{\cos 2\theta} \cdot 3\cos^2\theta \cdot (-\sin\theta) - \cos^3\theta \frac{1}{2\sqrt{\cos 2\theta}} \times (-2\sin 2\theta)}{\cos 2\theta}$$

$$= \frac{-3\sqrt{\cos 2\theta}\cos^2\theta \sin\theta + \frac{\cos^3\theta \cdot \sin 2\theta}{\sqrt{\cos 2\theta}}}{\cos 2\theta}$$

$$= \frac{-3\cos 2\theta \cos^2\theta \sin\theta + \cos^3\theta \cdot \sin 2\theta}{\sqrt{\cos 2\theta}\ \cos 2\theta}$$

$$\text{So, } \frac{dy}{dx} = \frac{-3\cos 2\theta \cos^2\theta \sin\theta + \cos^3\theta \cdot \sin 2\theta}{3\cos 2\theta \sin^2\theta \cos\theta + \sin^3\theta \cdot \sin 2\theta}$$

$$= \frac{\cos^2\theta \sin\theta (-3\cos 2\theta + 2\cos^2\theta)}{\sin^2\theta \cos\theta (3\cos 2\theta + 2\sin^2\theta)}$$

$$= \cot\theta \cdot \left(\frac{3 - 4\cos^2\theta}{3 - 4\sin^2\theta}\right) \quad \text{Ans}$$

(14) $x = a(\cos\theta + \ln\tan\theta/2)$,

$y = a\sin\theta$, when $\theta = \pi/3$

$$\therefore \frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\frac{\theta}{2}} \sec^2\frac{\theta}{2} \times \frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{2}\,\frac{1}{\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(-\sin\theta + \operatorname{cosec}\theta\right)$$

$$\& \ \frac{dy}{d\theta} = a\cos\theta$$

$$\text{So, } \frac{dy}{dx} = \frac{\cancel{a}\cos\theta}{\cancel{a}\left(-\sin\theta + \operatorname{cosec}\theta\right)}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\pi/3}{-\sin\pi/3 + \operatorname{cosec}\pi/3}$$

$$= \frac{1/2}{-\frac{\sqrt{3}}{2} + \frac{2}{\sqrt{3}}}$$

$$= \frac{1/2}{(4-3)} \times 2\sqrt{3} = \sqrt{3}$$

Ans

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(15) $\tan y = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\Rightarrow y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$ & $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$

Put $t = \tan\theta$

$\Rightarrow y = \tan^{-1}(\tan 2\theta)$ & $x = \sin^{-1}(\sin 2\theta)$

$\Rightarrow y = 2\theta$ & $x = 2\theta$

$\therefore \frac{dx}{d\theta} = 2$ & $\frac{dy}{d\theta} = 2$

So, $\frac{dy}{dx} = 1$ Ans

(16) $x = \sec^{-1}\left(\frac{1}{1-2t^2}\right)$, $y = \sin^{-1}(3t-4t^3)$

Put $x = \sin\theta$

$\Rightarrow x = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$, $y = \sin^{-1}(\sin 3\theta)$

$\Rightarrow x = 2\theta$, $y = 3\theta$

$\therefore \frac{dy}{dx} = \frac{3}{2}$ Ans

(17) $x = \sec^{-1}\left(\frac{1}{2t^2-1}\right), y = \tan^{-1}\left(\frac{t}{\sqrt{1-t^2}}\right)$

Put $t = \cos\theta$

$\Rightarrow x = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right), y = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

$\Rightarrow x = 2\theta, \quad y = \frac{\pi}{2} - \theta$

So, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-1}{2}$ Ans

(18) $x = \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right), y = \tan^{-1}\left(\frac{3t-t^3}{1-3t^2}\right)$

Put $t = \tan\theta$

$\Rightarrow x = 2\theta, \quad y = 3\theta$

So, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 3/2$

(19) $x = \sec^{-1}\left(\frac{1+t^2}{1-t^2}\right), y = \tan^{-1}\left(\frac{3t-t^3}{1-3t^2}\right)$

Put $t = \tan\theta$.

(20) $x = \cos^{-1}(8t^4 - 8t^2 + 1)$, $y = \sin^{-1}(3t - 4t^3)$

Put $t = \sin\theta$

$\Rightarrow x = \cos^{-1}\left(8\sin^2\theta(\sin^2\theta - 1) + 1\right)$

& $y = \sin^{-1}(\sin 3\theta)$

$\Rightarrow x = \cos^{-1}(1 - 8\sin^2\theta\cos^2\theta)$, $y = 3\theta$

$\Rightarrow x = \cos^{-1}(1 - 2\sin^2 2\theta)$, $y = 3\theta$

$\Rightarrow x = \cos^{-1}(\cos 4\theta)$, $y = 3\theta$

$\Rightarrow x = 4\theta$, $y = 3\theta$

So, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{3}{4}$ Ans

Derivative of one function with respect to another function.

To find the derivative of $f(x)$ w.r.t another function, say $g(x)$.

Let $y = f(x)$ & $z = g(x)$

$$\therefore \frac{dy}{dx} = f'(x) \quad \& \quad \frac{dz}{dx} = g'(x)$$

So, $\frac{dy}{dz}$ is the required derivative

Thus, $$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

(21) Derivative of $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ w.r.t $\tan^{-1}x$.

Let $y = \frac{\tan^{-1}x}{1+\tan^{-1}x}$ & $z = \tan^{-1}x$

~~$\therefore \frac{dy}{dx}$~~ Put $x = \tan\theta$

$\Rightarrow y = \frac{\theta}{1+\theta}$ & $z = \theta$

$$\therefore \frac{dy}{d\theta} = \frac{(1+\theta)\cdot 1 - \theta\cdot 1}{(1+\theta)^2}$$

$$= \frac{1}{(1+\theta)^2}$$

& $\frac{dz}{d\theta} = 1.$

$$\text{So, } \frac{dy}{dz} = \frac{\frac{1}{(1+\theta)^2}}{1}$$

$$= \frac{1}{(1+\theta)^2} = \frac{1}{(1+\tan^{-1}x)^2}$$

(22) Derivative of $x^{\sin^{-1}x}$ w.r.t $\sin^{-1}x$.

Let $y = x^{\sin^{-1}x}$ & $z = \sin^{-1}x$

$\Rightarrow \ln y = \sin^{-1}x \ln x$ & $\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right\}$$

$$= x^{\sin^{-1}x}\left\{\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right\}$$

So, $$\frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

$$= \frac{x^{\sin^{-1}x}\left(\sqrt{1-x^2}\sin^{-1}x + x\ln x\right)}{\cancel{x}\cancel{\sqrt{1-x^2}}\left(1/\cancel{\sqrt{1-x^2}}\right)}$$

$$= \frac{x^{\sin^{-1}x}\left(\sqrt{1-x^2}\sin^{-1}x + x\ln x\right)}{x}$$

Ans

(23) Find $\frac{dy}{dx}$ where (i) $y = \frac{(x+2)}{(x-1)(x+5)}$

(ii) $y = \sqrt{\frac{x^2+1}{x^2-1}}$

(iii) $y = \sqrt{\dfrac{1+x+x^2}{1-x+x^2}}$

(iv) $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$

(v) $y = \dfrac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$

(vi) $y = \dfrac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$

Solution

(i) $y = \dfrac{(x+2)}{(x-1)(x+5)}$

$\Rightarrow \ln y = \ln(x+2) - \ln(x-1)(x+5)$

$\Rightarrow \ln y = \ln(x+2) - \ln(x-1) - \ln(x+5)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x+2} - \dfrac{1}{x-1} - \dfrac{1}{x+5}$

$\Rightarrow \dfrac{dy}{dx} = y\left(\dfrac{1}{x+2} - \dfrac{1}{x-1} - \dfrac{1}{x+5}\right)$ Ans

(Put back the value of y).

(ii) $y = \sqrt{\dfrac{x^2+1}{x^2-1}}$

$\Rightarrow \ln y = \dfrac{1}{2} \ln\left(\dfrac{x^2+1}{x^2-1}\right)$

$\Rightarrow \ln y = \dfrac{1}{2} \ln(x^2+1) - \dfrac{1}{2} \ln(x^2-1)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{(x^2+1)} \times 2x - \dfrac{1}{2(x^2-1)} \times 2x$

$= \dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}$

$= \dfrac{x(x^2-1-x^2-1)}{x^4-1}$

$= \dfrac{-2x}{x^4-1}$

$\Rightarrow \dfrac{dy}{dx} = -2x\sqrt{\dfrac{x^2+1}{x^2-1}} \times \dfrac{1}{(x^4-1)}$

$= \dfrac{-2x\sqrt{x^2+1}}{\sqrt{x^2-1}\,(x^2-1)(x^2+1)}$

$= \dfrac{-2x}{\sqrt{x^2+1}\,(x^2-1)^{3/2}}$ Ans

(iii) $y = \sqrt{\dfrac{1+x+x^2}{1-x+x^2}}$

$\Rightarrow \ln y = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{2}\ln(1-x+x^2)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2x+1}{2(x^2+x+1)} - \dfrac{(2x-1)}{2(1-x+x^2)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2}\left\{\dfrac{2x+1}{x^2+x+1} - \dfrac{2x-1}{x^2-x+1}\right\}$

$= \dfrac{y}{2}\left\{\dfrac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}\right\}$

$= \dfrac{y}{\cancel{2}}\dfrac{\cancel{2}(1-x^2)}{(x^2+x+1)(x^2-x+1)}$

$= \dfrac{\sqrt{1+x+x^2}}{\sqrt{1-x+x^2}} \times \dfrac{(1-x^2)}{(1+x+x^2)(1-x+x^2)}$

$= \dfrac{(1-x^2)}{\sqrt{1+x+x^2}\cdot(1-x+x^2)^{3/2}}$ Ans

(iv) $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$

$\Rightarrow y = \sqrt{\dfrac{x^2-3x+2}{x^2-7x+12}}$

Same as (iii)

(v) $y = \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$

$= \dfrac{\left(\sqrt{1+x}+\sqrt{1-x}\right)^2}{(1+x)-(1-x)}$

$= \dfrac{1+x+1-x+2\sqrt{1-x^2}}{\cancel{1}+x-\cancel{1}+x}$

$= \dfrac{2+2\sqrt{1-x^2}}{2x}$

$= \dfrac{1+\sqrt{1-x^2}}{x}$

$\Rightarrow xy = 1+\sqrt{1-x^2}$

$$\therefore \frac{d}{dx}(xy) = \frac{d}{dx}\left(1+\sqrt{1-x^2}\right)$$

$$\Rightarrow x\frac{dy}{dx} + y = \frac{1}{\cancel{2}\sqrt{1-x^2}} \times (-\cancel{2}x)$$

$$\Rightarrow x\frac{dy}{dx} + y = \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \frac{y}{x}$$

$$= -\left(\frac{1}{\sqrt{1-x^2}} + \frac{1+\sqrt{1-x^2}}{x^2}\right)$$

$$= -\left(\frac{\cancel{x^2} + \sqrt{1-x^2} + 1 - \cancel{x^2}}{x^2\sqrt{1-x^2}}\right)$$

$$= -\left(\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}\right) \text{ Ans}$$

(vi) $$y = \frac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$$

$$= \frac{\left(\sqrt{a^2+x^2} + \sqrt{a^2-x^2}\right)^2}{(a^2+x^2)-(a^2-x^2)}$$

$$= \frac{a^2+\cancel{x^2}+a^2-\cancel{x^2}+2\sqrt{a^4-x^4}}{2x^2}$$

$$= \frac{2a^2+2\sqrt{a^4-x^4}}{2x^2}$$

$$= \frac{a^2+\sqrt{a^4-x^4}}{x^2}$$

$$\Rightarrow yx^2 = a^2 + \sqrt{a^4-x^4}$$

$$\therefore \frac{d}{dx}(yx^2) = \frac{d}{dx}\left(a^2+\sqrt{a^4-x^4}\right)$$

$$\Rightarrow x^2\frac{dy}{dx} + y \times 2x = \frac{-1}{2\sqrt{a^4-x^4}} \times 4x^3$$

$$\Rightarrow x^2\frac{dy}{dx} + 2xy = \frac{-2x^3}{\sqrt{a^4-x^4}}$$

$$\Rightarrow x^2\frac{dy}{dx} + 2xy = \frac{-2x^3}{(yx^2-a^2)}$$

$$\cancel{\Rightarrow x^2(yx^2-a^2)\frac{dy}{dx}}$$

$$\Rightarrow x^2\frac{dy}{dx} = \frac{-2x^3}{(yx^2-a^2)} - 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(yx^2-a^2)} - \frac{2y}{x}$$

$$= -2\left(\frac{x}{yx^2-a^2} + \frac{y}{x}\right)$$

$$= -2\left(\frac{x^2+y^2x^2-a^2y}{x(yx^2-a^2)}\right)$$

Ans

(24) If $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) + \sin(e^x)$

find $\frac{dy}{dx}$.

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) + \sin(e^x)$$

$$= \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right) + \sin e^x$$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cancel{2}\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right) + \sin e^x$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) + \sin e^x$$

$$= \frac{\pi}{4} - \frac{x}{2} + \sin e^x$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2} + \cos e^x \cdot e^x \quad \text{Ans}$$

(25) If $y = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

find $\frac{dy}{dx}$

Put $x = a\sin\theta$

$$\therefore y = \frac{a}{2}\sin\theta\sqrt{a^2(1-\sin^2\theta)} + \frac{a^2}{2}\sin^{-1}\left(\frac{a\sin\theta}{a}\right)$$

$$= \frac{a^2}{2}\sin\theta\cos\theta + \frac{a^2}{2}\theta$$

$$\Rightarrow y = \frac{a^2}{4}\sin 2\theta + \frac{a^2}{2}\theta$$

& $x = a\sin\theta$

$$\therefore \frac{dy}{d\theta} = \frac{2a^2\cos 2\theta}{4} + \frac{a^2}{2}$$

$$= \frac{a^2\cos 2\theta}{2} + \frac{a^2}{2}$$

& $\frac{dx}{d\theta} = a\cos\theta$

$$\text{So, } \frac{dy}{dx} = \frac{\frac{a^2}{2} + \frac{a^2\cos 2\theta}{2}}{a\cos\theta}$$

$$= \frac{a\cancel{2}(1+\cos 2\theta)}{2\cancel{a}\cos\theta} = \frac{\cancel{2}a\cos^2\theta}{\cancel{2}\cos\theta}$$

$$= a\cos\theta = a\sqrt{1-\frac{x^2}{a^2}}$$

$$= \sqrt{a^2-x^2} \quad \text{Ans}$$

(2e) If $y = \ln\left(x + \sqrt{x^2 - a^2}\right) + \sin^{-1}\left(\frac{x}{a}\right)$
find $\frac{dy}{dx}$

Put $x = a\sin\theta$

$\Rightarrow y = \ln\left(a\sin\theta + a\cos\theta\right) + \theta$

$\therefore \frac{dx}{d\theta} = a\cos\theta$

& $\frac{dy}{d\theta} = \frac{1}{a(\sin\theta + \cos\theta)} a \times (\cos\theta - \sin\theta) + 1$

$= \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} + 1$

$\therefore \frac{dy}{dx} = \frac{\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} + 1}{a\cos\theta}$

$= \frac{2\cos\theta}{a\cos\theta(\cos\theta + \sin\theta)}$

$= \frac{2}{a\left(\frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right)}$

$= \frac{2}{x + \sqrt{a^2 - x^2}}$ Ans

(27) If $y = \ln\left(\frac{1+x}{1-x}\right)^{1/4} - \frac{1}{2}\tan^{-1}(x)$

find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}\right) - \frac{1}{2}\frac{d}{dx}\tan^{-1}(x)$$

$$= \frac{1}{4}\frac{d}{dx}\left(\ln(1+x) - \ln(1-x)\right) - \frac{1}{2}\frac{1}{1+x^2}$$

$$= \frac{1}{4}\left(\frac{1}{1+x} - \frac{(-1)}{1-x}\right) - \frac{1}{2(1+x^2)}$$

$$= \frac{1}{4}\left(\frac{\cancel{1}-x+\cancel{1}+x}{1-x^2}\right) - \frac{1}{2(1+x^2)}$$

$$= \frac{1}{2(1-x^2)} - \frac{1}{2(1+x^2)}$$

$$= \frac{1}{2}\left\{\frac{1}{1-x^2} - \frac{1}{1+x^2}\right\}$$

$$= \frac{1}{2}\left(\frac{1+x^2-1+x^2}{1-x^4}\right)$$

$$= \frac{x^2}{1-x^4} \quad \text{Ans}$$

(28) If $y = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + a^x$, find $\frac{dy}{dx}$.

$$y = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + a^x$$

$$= \frac{1}{2}\ln\tan^2\frac{x}{2} + a^x$$

$$= \ln\left(\tan\frac{x}{2}\right) + a^x$$

$$\therefore \frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}}\sec^2\frac{x}{2}\times\frac{1}{2} + a^x\ln a$$

$$= \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} + a^x\ln a$$

$$= \frac{1}{\sin x} + a^x\ln a$$

$$= \operatorname{cosec} x + a^x\ln a$$

Ans

(29) If $y = \log_{\sin x} \sec x + 10^{x^2}$, find $\frac{dy}{dx}$

$$y = \log_{\sin x} e \times \log_e \sec x + 10^{x^2}$$

$$= \frac{\ln \sec x}{\ln \sin x} + 10^{x^2}$$

$$\therefore \frac{dy}{dx} = \frac{\ln \sin x \cdot \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x - \ln \sec x \cdot \frac{\cos x}{\sin x}}{(\ln \sin x)^2} + 10^{x^2} \ln 10 \times 2x$$

$$= \frac{\tan x \ln \sin x - \cot x \ln \sec x}{(\ln \sin x)^2} + 10^{x^2} \ln 10 \times 2x$$

Ans

(30) If $y = 2x\tan^{-1}x - \ln(1+x^2)$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2\left\{\frac{x}{1+x^2} + \tan^{-1}x \cdot 1\right\} - \frac{1}{1+x^2} \times 2x$$

$$= \frac{2x}{1+x^2} + 2\tan^{-1}x - \frac{2x}{1+x^2}$$

$$= 2\tan^{-1}x$$ Ans

(31) If $y = \frac{1}{3}\ln\left(\frac{x+1}{\sqrt{x^2-x+1}}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

find $\frac{dy}{dx}$.

$$y = \frac{1}{3}\left\{\ln(x+1) - \ln\left(\sqrt{x^2-x+1}\right)\right\} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{3(x+1)} - \frac{(2x-1)}{6(x^2-x+1)} + \frac{1}{\sqrt{3}}\frac{1}{1+\frac{(2x-1)^2}{3}} \times \frac{1}{\sqrt{3}} \times 2$$

$$= \frac{1}{3(x+1)} - \frac{2x-1}{6(x^2-x+1)} + \frac{2}{\cancel{3}} \times \frac{\cancel{3}}{(3+(2x-1)^2)}$$

$$= \frac{2x^2-2x+2-(2x^2+2x-x-1)}{6(x+1)(x^2-x+1)} + \frac{2}{3+4x^2+1-4x}$$

$$= \frac{-3x+3}{6(x+1)(x^2-x+1)} + \frac{2}{4x^2-4x+4}$$

$$= \frac{\cancel{3}(1-x)}{\cancel{6}_{2}(x+1)(x^2-x+1)} + \frac{\cancel{2}}{\cancel{4}_{2}(x^2-x+1)}$$

$$= \frac{1}{2(x^2-x+1)}\left\{\frac{1-x}{(x+1)} + 1\right\}$$

$$= \frac{1}{2(x^2-x+1)} \times \frac{1-\cancel{x}+\cancel{x}+1}{(x+1)}$$

$$= \frac{\cancel{2}\,1}{\cancel{2}(x^2-x+1)(x+1)}$$

$$= \frac{1}{(x+1)(x^2-x+1)} \quad \text{Ans}$$

$$= \frac{1}{x^3+1}$$

(32) If $y = \sqrt{1+\sin x} + \sqrt{\frac{1-x^2}{1+x^2}}$

find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+\sin x}} \cdot \cos x + \frac{1}{2\sqrt{\frac{1-x^2}{1+x^2}}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{\cos x}{2\sqrt{1+\sin x}} + \frac{1}{2}\sqrt{\frac{1+x^2}{1-x^2}}\,\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{\cos x}{2\sqrt{1+\sin x}} + \frac{1}{2}\sqrt{\frac{1+x^2}{1-x^2}}\left(\frac{(1+x^2)\times(-2x)-(1-x^2)\times 2x}{(1+x^2)^2}\right)$$

$$= \frac{\cos x}{2\sqrt{1+\sin x}} + \frac{1}{2}\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} \times \frac{(-4x)}{(1+x^2)^2}$$

$$= \frac{\cos x}{2\sqrt{1+\sin x}} - \frac{2x}{\sqrt{1-x^2}\sqrt{1+x^2}\,(1+x^2)}$$

$$= \frac{\cos x}{2\sqrt{1+\sin x}} - \frac{2x}{\sqrt{1-x^4}\,(1+x^2)}$$

Ans

(33) If $y = e^{x/y}$, the show that $\frac{dy}{dx} = \frac{y}{(y+x)}$

$$y = e^{x/y}$$

$$\Rightarrow \ln y = \ln e^{x/y}$$

$$\Rightarrow \ln y = \frac{x}{y}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{y.1 - x\frac{dy}{dx}}{y^2}$$

$$\Rightarrow y\frac{dy}{dx} = y - x\frac{dy}{dx}$$

$$\Rightarrow (y+x)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{y+x}$$ Proved

(34) If $y = e^{y/x}$, Then show that $\frac{dy}{dx} = \frac{y^2}{x(y-x)}$

Same as (33).

(35) If $y\sqrt{1+x^2} = x$, the show that

$$x^3 \frac{dy}{dx} = y^3.$$

Given, $y\sqrt{1+x^2} = x$

$$\Rightarrow y^2(1+x^2) = x^2$$

$$\therefore \frac{d}{dx}\left(y^2(1+x^2)\right) = 2x$$

$$\Rightarrow (1+x^2)\, 2y \frac{dy}{dx} + y^2 \times 2x = 2x$$

$$\Rightarrow y(1+x^2)\frac{dy}{dx} = x(1-y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(1-y^2)}{y(1+x^2)}$$

$$\Rightarrow x^3 \frac{dy}{dx} = \frac{x^4(1-y^2)}{y(1+x^2)}$$

$$\left(\because x^2 = y^2(1+x^2)\right) = \frac{y^4(1+x^2)^2(1-y^2)}{y(1+x^2)}$$

$$\Rightarrow \frac{dy}{dx} = y^3(1+x^2)(1-y^2)$$

$$= y^3(1+x^2-y^2-x^2y^2)$$

$$= y^3(1+x^2-(x^2y^2+y^2))$$

$$= y^3(1+x^2-x^2) \quad \because y^2+y^2x^2 = x^2$$

$$= y^3 \quad \text{Proved}$$

(36) If $y^x = e^{y-x}$, then show that

$$\frac{dy}{dx} = \frac{(\ln(ey))^2}{\ln y}$$

$$y^x = e^{y-x}$$

$$\Rightarrow x\ln y = (y-x)\ln e$$

$$\therefore \frac{d}{dx}(x\ln y) = \frac{d}{dx}(y-x)$$

$$\Rightarrow \frac{x}{y}\frac{dy}{dx} + \ln y = \frac{dy}{dx} - 1$$

$$\Rightarrow \frac{x-y}{y}\frac{dy}{dx} = -(1+\ln y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+\ln y)}{y-x}$$

$$= \frac{y(\ln e + \ln y)}{\frac{x \ln y}{\ln e}}$$

$\left(\because y - x = \frac{x \ln y}{\ln e}\right)$

$$= \frac{y(\ln ey) \times \ln e}{x \ln y}$$

Note $y - x = x \frac{\ln y}{\ln e}$

$$\Rightarrow \frac{y}{x} - 1 = \frac{\ln y}{\ln e}$$

$$\Rightarrow \frac{y}{x} = \frac{\ln y + \ln e}{\ln e}$$

$$= \frac{\ln(ey)}{\ln e}$$

$$\therefore \frac{dy}{dx} = \frac{\ln(ey) \cdot \ln(ey) \cancel{\times \ln e}}{\cancel{\ln e} \times \ln y}$$

$$= \frac{(\ln(ey))^2}{\ln y}$$ Proved

(37) If $y = \dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$, Then show that $\dfrac{dy}{d\theta} + \dfrac{1}{1-\cos\theta} = 0$.

$$y = \frac{1+\cos\theta+\sin\theta}{1-\cos\theta+\sin\theta}$$

$$= \frac{2\cos^2\theta/2 + 2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{2\cos\theta/2\,\cancel{(\cos\theta/2+\sin\theta/2)}}{2\sin\theta/2\,\cancel{(\sin\theta/2+\cos\theta/2)}}$$

$$= \cot\theta/2$$

$$\therefore \frac{dy}{d\theta} = -\frac{1}{2}\operatorname{cosec}^2\theta/2 = \frac{-1}{2\sin^2\theta/2}$$

$$\Rightarrow \frac{dy}{d\theta} + \frac{1}{2\sin^2\theta/2} = 0$$

$$\Rightarrow \frac{dy}{d\theta} + \frac{1}{1-\cos\theta} = 0 \quad \text{Proved}$$

(38) If $\tan y = \dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$,

Then show that $\dfrac{dy}{dx} = 1/2$

$$\tan y = \frac{\frac{\sin x}{\cos x} + \frac{1}{\cos x} - 1}{\frac{\sin x}{\cos x} - \frac{1}{\cos x} + 1}$$

$$= \frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x}$$

$$= \frac{2\sin x/2 \cos x/2 + 2\sin^2 x/2}{2\sin x/2 \cos x/2 - 2\sin^2 x/2}$$

$$= \frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}$$

$$= \frac{1 + \tan x/2}{1 - \tan x/2}$$

$$= \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Proved