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FIRST ORDER DERIVATIVE

1. If $y = f(\ln x)$ $find \frac{dy}{dx}$.

$$Solution: \frac{dy}{dx} = \frac{d}{dx}\left(f(\ln x)\right) = f'(\ln x) \times \frac{d}{dx} \ln x$$

$$f'(\ln x) \times \frac{1}{x} = \frac{f'(\ln x)}{x}$$

2. If $y = \ln\left(f(x)\right)$ $find \frac{dy}{dx}$.

$$Solution: \frac{dy}{dx} = \frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{1}{f(x)} \times \frac{d}{dx} f(x)$$

$$\frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

3. If $y = ce^{\frac{x}{x-a}}$ $find \frac{dy}{dx}$.

$$Solution: \frac{dy}{dx} = \frac{d}{dx}\left(ce^{\frac{x}{x-a}}\right) = c \times \frac{d}{dx} e^{\frac{x}{x-a}}$$

$$= ce^{\frac{x}{x-a}} \times \frac{d}{dx}\left(\frac{x}{x-a}\right)$$

$$= ce^{\frac{x}{x-a}} \times \left(\frac{(x-a) \times \frac{d}{dx} x - x \times \frac{d}{dx}(x-a)}{(x-a)^2}\right)$$

$$= ce^{\frac{x}{x-a}} \times \left(\frac{(x-a) \times 1 - x \times 1}{(x-a)^2}\right)$$

$$= ce^{\frac{x}{x-a}} \times \left(\frac{(x-a) - x}{(x-a)^2}\right)$$

$$= -\frac{ace^{\frac{x}{x-a}}}{(x-a)^2} = -\frac{ay}{(x-a)^2}$$

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4. $If\ f(x) = 3e^{x^2}, find\ the\ value\ of\ f'(x) - 2xf(x) + \frac{1}{3}f(0) - f'(0).$

$$Solution:\quad f'(x) = \frac{d}{dx}3e^{x^2} = 3e^{x^2} \times \frac{d}{dx}x^2$$
$$= 3e^{x^2} \times 2x = 6xe^{x^2}$$

$$\therefore f'(0) = 0$$

$$now,\qquad 2xf(x) = 2x \times 3e^{x^2} = 6xe^{x^2}$$

$$also,\qquad \frac{1}{3}f(0) = \frac{1}{3}3e^0 = 1$$

$$\therefore f'(x) - 2xf(x) + \frac{1}{3}f(0) - f'(0) = 6xe^{x^2} - 6xe^{x^2} + 1 - 0 = 1$$

5. $If\ y = f(x)^{\varphi(x)}, find\ \frac{dy}{dx}$

$$y = f(x)^{\varphi(x)}$$

$$\Rightarrow \ln y = \ln f(x)^{\varphi(x)} = \varphi(x)\ln f(x)$$

$$now\ differentiating\ both\ side\ w.r.t\ 'x'$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\varphi(x)\ln f(x))$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \varphi(x)\frac{d}{dx}\ln f(x) + \ln f(x)\frac{d}{dx}\varphi(x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\varphi(x)f'(x)}{f(x)} + \varphi'(x)\ln f(x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = f(x)^{\varphi(x)}\left\{\frac{\varphi(x)f'(x)}{f(x)} + \varphi'(x)\ln f(x)\right\}$$

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6. $If\ f(x) = x + 1, find \frac{d}{dx}\big(\ fof(x)\big)$

$$fof(x) = f\big(f(x)\big) = f(x+1) = (x+1) + 1 = x + 2$$

$$\therefore \frac{d}{dx}\big(\ fof(x)\big) = \frac{d}{dx}(x+2) = 1$$

7. Find the derivatives of the following functions
   i. $y = (x+15)^9$
   ii. $y = \frac{1}{\sqrt{ax+b}}$
   iii. $y = \sqrt{ax^2 + bx + c}$
   iv. $y = \frac{1}{\sqrt{ax^2+bx+c}}$
   v. $y = \sqrt{x} - \frac{1}{\sqrt{x}}, x > 0$
   vi. $y = \sqrt{\frac{1+x}{1-x}}$
   vii. $y = \sin x^o$
   viii. $y = \tan(ax+b)$
   ix. $y = \tan\{f(x)\}$
   x. $y = \sin\{f(x)\}^2$
   xi. $y = \frac{\cos x}{1+\sin x}$
   xii. $y = \frac{\cos x - \cos 2x}{1-\cos x}$

Solution i.

$$\frac{dy}{dx} = \frac{d}{dx}(x+15)^9$$

$$\Rightarrow \frac{dy}{dx} = 9(x+15)^8 \times \frac{d}{dx}(x+15)$$

$$\Rightarrow \frac{dy}{dx} = 9(x+15)^8$$

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Solution ii.

$$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{\sqrt{ax+b}} = \frac{d}{dx}(ax+b)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}(ax+b)^{-\frac{1}{2}-1} \times \frac{d}{dx}(ax+b)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}(ax+b)^{-\frac{3}{2}} \times a$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}a\frac{1}{(ax+b)^{\frac{3}{2}}} = -\frac{a}{2(ax+b)\sqrt{ax+b}}$$

Solution iii.

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{ax^2+bx+c} = \frac{d}{dx}(ax^2+bx+c)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(ax^2+bx+c)^{\frac{1}{2}-1} \times \frac{d}{dx}(ax^2+bx+c)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(ax^2+bx+c)^{-\frac{1}{2}} \times (2ax+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{2ax+b}{(ax^2+bx+c)^{\frac{1}{2}}} = -\frac{2ax+b}{2\sqrt{ax^2+bx+c}}$$

Solution v.

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x} - \frac{d}{dx}\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} - \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

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$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} = \frac{1}{2\sqrt{x}}\left(1 + \frac{1}{x}\right)$$

$$\boldsymbol{Important\ Formula:\ \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$$

$$\boldsymbol{and\ \ \frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2x\sqrt{x}}}$$

Solution vi.

Put $\frac{1+x}{1-x} = z$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\sqrt{z} = \frac{1}{2\sqrt{z}}\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1+x}{1-x}}}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \left\{\frac{(1-x) \times \frac{d}{dx}(1+x) - (1+x) \times \frac{d}{dx}(1-x)}{(1-x)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \left\{\frac{(1-x) \times 1 - (1+x) \times (-1)}{(1-x)^2}\right\}$$

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$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \left\{\frac{1-x+1+x}{(1-x)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \left\{\frac{2}{(1-x)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-x}{1+x}} \times \left\{\frac{1}{(1-x)^2}\right\} = \frac{1}{(1-x)(1-x)}\sqrt{\frac{1-x}{1+x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1-x)\sqrt{1-x}}\frac{1}{\sqrt{1+x}} = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$\boldsymbol{Important\ Differentials}: \frac{d}{dx}\left(\frac{1+x}{1-x}\right) = \frac{2}{(1-x)^2}$$

$$\boldsymbol{Similarly}\ \frac{d}{dx}\left(\frac{1-x}{1+x}\right) = \frac{-2}{(1+x)^2}$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{-4x}{(1+x^2)^2}$$

$$\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right) = \frac{4x}{(1-x^2)^2}$$

You can prove these easily by using the division rule.

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Solution vii.

$$y = \sin x^o$$

$$\Rightarrow y = \sin\frac{\pi x}{180} \ (\textit{convert to radians})$$

$$\Rightarrow \frac{dy}{dx} = \cos\frac{\pi x}{180} \times \frac{d}{dx}\frac{\pi x}{180}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\cos\frac{\pi x}{180}$$

Solution ix/ viii.

$$y = \tan f(x)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 f(x) \times \frac{d}{dx} f(x)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 f(x) \times f'(x) = f'(x)\sec^2 f(x)$$

For problem viii. put $f(x) = ax + b$ *and see that* $f'(x) = a$

Solution x.

$$y = \sin\{f(x)\}^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin\{f(x)\}^2$$

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$$\Rightarrow \frac{dy}{dx} = \cos\{f(x)\}^2 \times \frac{d}{dx} f(x)^2$$

$$\Rightarrow \frac{dy}{dx} = \cos\{f(x)\}^2 \times 2f(x) \frac{d}{dx} f(x)$$

$$\Rightarrow \frac{dy}{dx} = \cos\{f(x)\}^2 \times 2f(x) \times f'(x)$$

Solution xi.

$$y = \frac{\cos x}{1 + \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \frac{\cos x}{1 + \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x) \times \frac{d}{dx} \cos x - \cos x \ \times \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x) \times (-\sin x) - \cos x \ \times (\cos x)}{(1 + \sin x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} = -\frac{\sin x}{(1 + \sin x)^2}$$

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Solution xii.

$$y = \frac{\cos x - \cos 2x}{1 - \cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x - \cos 2x}{1 - \cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 - \cos x) \times \frac{d}{dx}(\cos x - \cos 2x) - (\cos x - \cos 2x) \times \frac{d}{dx}(1 - \cos x)}{(1 - \cos x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 - \cos x) \times (-\sin x + 2\sin 2x) - (\cos x - \cos 2x) \times (\sin x)}{(1 - \cos x)^2}$$

$$= \frac{-\sin x + \sin x \cos x + 2\sin 2x - 2\cos x \sin 2x - \sin x \cos x + \sin x \cos 2x}{(1 - \cos x)^2}$$

$$= \frac{-\sin x + 4\sin x \cos x - 4\cos^2 x \sin x + \sin x \cos 2x}{(1 - \cos x)^2}$$

$$= \frac{\sin x\,(-1 + 4\cos x - 4\cos^2 x + \cos 2x)}{(1 - \cos x)^2}$$

$$= \frac{\sin x\,(-1 + 4\cos x - 4\cos^2 x + 2\cos^2 x - 1)}{(1 - \cos x)^2}$$

$$= \frac{\sin x\,(-2\cos^2 x + 4\cos x - 2)}{(1 - \cos x)^2}$$

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$$= \frac{-2\sin x\,(\cos^2 x - 2\cos x + 1)}{(1-\cos x)^2}$$

$$= \frac{-2\sin x\,(1-\cos x)^2}{(1-\cos x)^2}$$

$$= -2\sin x$$

8. Find the derivatives of the following functions
   i. $y = e^{9x}$
   ii. $y = e^{2x^3+3x^2+4x+5}$
   iii. $y = e^{\sin^2 x}$
   iv. $y = 5^{x\log_5 7}$

$Solution\ i - iii.$

$$if\ y = e^{f(x)}$$

$$\Rightarrow \frac{dy}{dx} = e^{f(x)} \times f'(x)$$

$$i. \qquad f(x) = 9x, therefore\ f'(x) = 9$$

$$\Rightarrow \frac{dy}{dx} = e^{9x} \times 9 = 9e^{9x}$$

$$ii. \qquad f(x) = 2x^3 + 3x^2 + 4x + 5, therefore$$
$$f'(x) = 6x^2 + 6x + 4$$

$$\Rightarrow \frac{dy}{dx} = e^{2x^3+3x^2+4x+5} \times (6x^2 + 6x + 4)$$
$$= (6x^2 + 6x + 4)e^{2x^3+3x^2+4x+5}$$

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$iii.$ $\quad f(x) = \sin^2 x\,, therefore\ f'(x) = 2\sin x \cos x$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^2 x} \times 2\sin x \cos x = \sin 2x\, e^{\sin^2 x}$$

$Solution\ iv.$

$$y = 5^{x log_5 7}$$

$$\Rightarrow y = 5^{\log_5 7^x}$$

$$\Rightarrow y = 7^x$$

$$\ln y = \ln 7^x = x \ln 7$$

Now differentiate both side w.r.t 'x'

$$\frac{1}{y}\frac{dy}{dx} = \ln 7$$

$$\frac{dy}{dx} = y \ln 7 = 7^x \ln 7$$

9. Find the derivatives of the following functions
   i. $y = \ln(ax^2 + bx + c)$
   ii. $y = \ln(\tan(2x + 3))$
   iii. $y = \sqrt{\ln(ax + b)}$
   iv. $y = \ln(\ln x)$
   v. $y = \ln(\sin(\log x))$
   vi. $y = \ln(\sec x + \tan x)$

Using chain rule see that if

$$\boldsymbol{y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)}$$

$Solution\ i.$

$$y = \ln(ax^2 + bx + c)$$

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$$\Rightarrow \frac{dy}{dx} = \frac{1}{ax^2 + bx + c} \times \frac{d}{dx}(\, ax^2 + bx + c)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ax + b}{ax^2 + bx + c}$$

Solution ii.

$$y = \ln(\tan(2x + 3))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\tan(2x + 3)} \times \frac{d}{dx}(\, \tan(2x + 3))$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2(2x + 3)}{\tan(2x + 3)} \times \frac{d}{dx}(\, 2x + 3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sec^2(2x + 3)}{\tan(2x + 3)}$$

Solution iii.

$$y = \sqrt{\ln(ax + b)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\ln(ax + b)}} \times \frac{d}{dx}(\, \ln(ax + b))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\ln(ax + b)}} \times \frac{1}{ax + b}\frac{d}{dx}(ax + b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{2(ax + b)\sqrt{\ln(ax + b)}}$$

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Solution iv.

$$y = \ln(\ln x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln x} \times \frac{d}{dx}(\ln x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x \ln x}$$

Solution v.

$$y = \ln(\sin(\ln x))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin(\ln x)} \times \frac{d}{dx}(\sin(\ln x))$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(\ln x)}{\sin(\ln x)} \times \frac{d}{dx}(\ln x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cot(\ln x)}{x}$$

Solution vi.

$$y = \ln(\sec x + \tan x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \frac{d}{dx}(\sec x + \tan x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x}$$

$$\Rightarrow \frac{dy}{dx} = \sec x$$

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10. From the given parametric equations, find the differential coefficient.

i. $x = a\cos t\ , y = b\sin t$

ii. $x = at^2\ , y = 2at$

iii. $x = a(t - \sin t)\ , y = a(1 - \cos t)$

iv. $x = t^2 + 2t\ , y = t^3 - 3t$

General approach for parametric equations:

$$if\ x = \varphi(t)\ and\ y = \omega(t)$$

$$\Rightarrow \frac{dx}{dt} = \varphi'(t)\ and\ \frac{dy}{dt} = \omega'(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\omega\prime(t)}{\varphi'(t)}$$

Solution i.

$$\frac{dy}{dx} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t$$

Solution ii.

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

Solution iii.

$$\frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

Solution iv.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t + 2} = \frac{3(t-1)(t+1)}{2(t+1)} = \frac{3}{2}(t-1)$$

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11. Find the differential coefficients of the following functions

i. $y^2 - 4ax = 0$

ii. $x^2 + y^2 = a^2$

iii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

iv. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

v. $ax^2 + 2hxy + by^2 = 1$

vi. $\log_e(xy) = x^2 + y^2$

$$For\ functions\ of\ the\ form\ f(x,y) = c, where$$

$$y\ can\ or\ cannot\ be\ explicitly\ written\ in\ terms\ of\ x,$$

$$to\ find\ \frac{dy}{dx}\ differentiate\ both\ side\ w.r.t\ 'x'$$

$$and\ then\ make\ \frac{dy}{dx}\ the\ subject.$$

Solution i.

$$\frac{d}{dx}(y^2 - 4ax) = \frac{d}{dx}(0)$$

$$\Rightarrow 2y\frac{dy}{dx} - 4a = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Solution ii.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

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Solution iii.

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

Solution iv.

$$\frac{d}{dx}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=\frac{d}{dx}(a^{\frac{2}{3}})$$

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Solution v.

$$\frac{d}{dx}(ax^2+2hxy+by^2)=\frac{d}{dx}(1)$$

$$\Rightarrow 2ax+2h\frac{d}{dx}(xy)+2by\frac{dy}{dx}=0$$

$$\Rightarrow ax+h\left(x\frac{dy}{dx}+y\right)+by\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}(hx+by)=-(ax+hy)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(ax+hy)}{(hx+by)}$$

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Solution vi.

$$\frac{d}{dx}(\ln xy) = \frac{d}{dx}(x^2 + y^2)$$

$$\Rightarrow \frac{1}{xy}\frac{d}{dx}(xy) = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - 2y\right) = 2x - \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1 - 2y^2}{y}\right) = \frac{2x^2 - 1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(2x^2 - 1)}{x(1 - 2y^2)}$$

12. $If\ f(x) = \ln x\,, find\ the\ derivative\ of\ f(\ln x\ ).$

Solution:

$$f(\ln x) = \ln(\ln x)$$

$$\Rightarrow \frac{d}{dx}\ f(\ln x) = \frac{d}{dx}\ln\ln x$$

$$\Rightarrow \frac{f'(\ln x)}{x} = \frac{1}{x\ln x}$$

$$\Rightarrow f'(\ln x) = \frac{1}{\ln x}$$

13. $If\ y = \ln\left(\tan\frac{x}{2}\right), find\ \frac{dy}{dx}$

Solution:

$$y = \ln\left(\tan\frac{x}{2}\right)$$

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$$\Rightarrow \frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}} \frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{\sin x} = cosec\ x$$

14. $If\ y = \ln(x + \sqrt{x^2 \pm a^2})\,, find\ \frac{dy}{dx}$

$$y = \ln(x + \sqrt{x^2 \pm a^2})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 \pm a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 \pm a^2}}\left(1 + \frac{1}{2\sqrt{x^2 \pm a^2}} \frac{d}{dx}(\,x^2 \pm a^2)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 \pm a^2}}\left(1 + \frac{2x}{2\sqrt{x^2 \pm a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 \pm a^2}}\left(\frac{x + \sqrt{x^2 \pm a^2}}{\sqrt{x^2 \pm a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 \pm a^2}}$$

15. $If\ y = \log_x e\,, then\ show\ that\ x\frac{dy}{dx} + y^2 = 0.$

Solution:

$$y = \log_x e = \frac{1}{\log_e x} = (\log_e x)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{(\log_e x)^2} \times \frac{d}{dx}(\log_e x)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x(\log_e x)^2}$$

$$\Rightarrow x\frac{dy}{dx} = -(\log_x e)^2 = -y^2$$

$$\Rightarrow x\frac{dy}{dx} + y^2 = 0$$

16. $if\ x^y = e^x\ then\ find\ \frac{dy}{dx}$

Solution:

$$x^y = e^x$$

$$\Rightarrow \ln x^y = \ln e^x$$

$$\Rightarrow y\ln x = x\ln e = x$$

$$\Rightarrow \frac{d}{dx}(y\ln x) = \frac{d}{dx}x$$

$$\Rightarrow \frac{y}{x} + \ln x\frac{dy}{dx} = 1$$

$$\Rightarrow \ln x\frac{dy}{dx} = \frac{x-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x\ln x}$$

Prime Maths

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Prime Maths

Find $\frac{dy}{dx}$ for the following functions.

i. $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$, $a \in \mathbb{R}$

ii. $y = \ln\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

iii. $y = \ln\left(\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\right)$

iv. $y = \ln\left(e^x \left(\frac{x-1}{x+1}\right)^{3/2}\right)$

v. $y = \ln\left\{(2x+4) + \sqrt{4x^2 + 16x - 12}\right\}$

vi. $y = \ln\left(\sqrt{x-a} + \sqrt{x-b}\right)$

vii. $y = \ln\left(\sqrt{\frac{a\cos x - b\sin x}{a\cos x + b\sin x}}\right)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution (i)

$$y = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \times \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$

$$= \frac{1}{\left(x + \sqrt{x^2 + a^2}\right)}\left(1 + \frac{1}{2\sqrt{x^2 + a^2}} \times \frac{d}{dx}(x^2 + a^2)\right)$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right)$$

$$= \frac{\cancel{\left(x+\sqrt{x^2+a^2}\right)}}{\cancel{\left(x+\sqrt{x^2+a^2}\right)}\left(\sqrt{x^2+a^2}\right)} = \frac{1}{\sqrt{x^2+a^2}}$$ Ans

(ii) $$y = \ln\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$= \frac{1}{2}\left\{\ln(1-\cos x) - \ln(1+\cos x)\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{1-\cos x}\times\frac{d}{dx}(1-\cos x) - \frac{1}{(1+\cos x)}\times\frac{d}{dx}(1+\cos x)\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right\}$$

$$= \frac{\cancel{2}\sin x}{\cancel{2}(1-\cos^2 x)} = \frac{\sin x}{\sin^2 x}$$

$$= \operatorname{cosec} x.$$ Ans

(iii) $y = \ln\left(\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

$$= \ln\left\{\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right\}$$

$$= \ln\left\{\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)^2}{1+x-1+x}\right\}$$

$$= \ln\left\{\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)^2}{2x}\right\}$$

$$= \ln\left\{\frac{1+\cancel{x}+1-\cancel{x}+2\sqrt{1-x^2}}{2x}\right\}$$

$$= \ln\left\{\frac{2+2\sqrt{1-x^2}}{2x}\right\}$$

$$= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$= \ln\left(1+\sqrt{1-x^2}\right) - \ln x$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+\sqrt{1-x^2}}\left(0+\frac{1}{2\sqrt{1-x^2}}\times(-2x)\right) - \frac{1}{x}$$

$$= \frac{-x}{1-x^2+\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{\cancel{-x^2}-1+\cancel{x^2}-\sqrt{1-x^2}}{x\left(1-x^2+\sqrt{1-x^2}\right)}$$

$$= \frac{-\cancel{(1+\sqrt{1-x^2})}}{x\cancel{(1+\sqrt{1-x^2})}\sqrt{1-x^2}}$$

$$= \frac{-1}{x\sqrt{1-x^2}} \quad \text{Ans}$$

(iv) $$y = \ln\left(e^x\left(\frac{x-1}{x+1}\right)^{3/2}\right)$$

$$= \ln e^x + \ln\left(\frac{x-1}{x+1}\right)^{3/2}$$

$$\Rightarrow y = x + \frac{3}{2}\left\{\ln(x-1) - \ln(x+1)\right\}$$

$$\therefore \frac{dy}{dx} = 1 + \frac{3}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$= 1 + \frac{3}{2}\left(\frac{x+1-x+1}{x^2-1}\right)$$

$$= 1 + \frac{3}{x^2-1} = \frac{x^2-1+3}{x^2-1}$$

$$= \frac{x^2+2}{x^2-1} \quad \text{Ans}$$

(v) $y = \ln\left\{(2x+4) + \sqrt{4x^2+16x-12}\right\}$

$= \ln\left\{(2x+4) + \sqrt{(2x+4)^2 - 28}\right\}$

Put $2x+4 = t$

$\therefore y = \ln\left\{t + \sqrt{t^2-28}\right\}$

diff^n both side w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{t+\sqrt{t^2-28}}\left(1 + \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2-28}}\right) \times \frac{dt}{dx}$$

$$= \frac{1}{\sqrt{t^2-28}} \times 2 \quad (\because t = 2x+4)$$

$$= \frac{2}{\sqrt{(2x+4)^2-28}} = \frac{2}{\sqrt{4x^2+16x-12}}$$

$$= \frac{1}{\sqrt{x^2+4x-3}}$$ Ans

(vi) $y = \ln\left(\sqrt{x-a} + \sqrt{x-b}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\sqrt{x-a}+\sqrt{x-b}\right)} \times \frac{d}{dx}\left(\sqrt{x-a} + \sqrt{x-b}\right)$$

$$= \frac{1}{\left(\sqrt{x-a} + \sqrt{x-b}\right)}\left(\frac{1}{2\sqrt{x-a}} + \frac{1}{2\sqrt{x-b}}\right)$$

$$= \frac{1}{2}\,\frac{\cancel{\left(\sqrt{x-a}+\sqrt{x-b}\right)}}{\cancel{\left(\sqrt{x-a}+\sqrt{x-b}\right)}\sqrt{x-a}\cdot\sqrt{x-b}}$$

$$= \frac{1}{2\sqrt{x-a}\,\sqrt{x-b}} \quad \text{Ans}$$

(vii) $y = \ln\left(\sqrt{\dfrac{a\cos x - b\sin x}{a\cos x + b\sin x}}\right)$

$$= \frac{1}{2}\ln\left(\frac{a\cos x - b\sin x}{a\cos x + b\sin x}\right)$$

$$= \frac{1}{2}\left\{\ln(a\cos x - b\sin x) - \ln(a\cos x + b\sin x)\right\}$$

differentiating both sides w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{2}\left\{\frac{-a\sin x - b\cos x}{a\cos x - b\sin x} - \frac{(b\cos x - a\sin x)}{a\cos x + b\sin x}\right\}$$

$$= \frac{1}{2}\,\frac{-(a\sin x + b\cos x)(a\cos x + b\sin x) - (a\cos x - b\sin x)(b\cos x - a\sin x)}{a^2\cos^2 x - b^2\sin^2 x}$$

$$= \frac{-2ab(\sin^2 x + \cos^2 x)}{2(a^2\cos^2 x - b^2\sin^2 x)}$$

$$= \frac{-ab}{a^2\cos^2 x - b^2\sin^2 x} \quad \text{Ans}$$

Find $\frac{dy}{dx}$ of the following functions.

i. $y = \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right), \; -1 < x < 1$

ii. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \; 0 < x < 1$

iii. $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

iv. $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \quad 0 < x < \pi$

v. $y = \tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right), \quad 0 < x < \pi$

vi. $y = \cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

vii. $y = \sin^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right)$

viii. $y = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

Solution

(i) $y = \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$\therefore y = \frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \frac{1}{2}\sin^{-1}(\sin 2\theta)$

$\Rightarrow y = \frac{1}{2} \times 2\theta = \theta = \tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$ Ans

(ii) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$\therefore y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$y = 2\theta = 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$ Ans

(iii) $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Put $x = \tan\theta$.

$\therefore y = \tan^{-1}(\tan 3\theta)$

$= 3\theta = 3\tan^{-1}x$.

$\Rightarrow \frac{dy}{dx} = \frac{3}{1+x^2}$ Ans

(iv) $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$

$$= \tan^{-1}\left(\frac{\cos^2 x/2 - \sin^2 x/2}{\cos^2 x/2 + \sin^2 x/2 - 2\cos x/2 \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{(\cos x/2 - \sin x/2)(\cos x/2 + \sin x/2)}{(\cos x/2 - \sin x/2)^2}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{1 + \tan x/2}{1 - \tan x/2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4} + x/2$$

$$\therefore y = \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$ Ans

(v) $y = \tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\cos^2 x/2}{2\sin^2 x/2}}\right)$

$= \tan^{-1}\left(\sqrt{\cot^2 x/2}\right)$

$= \tan^{-1}\left(\cot x/2\right)$ $(\because \cot x/2 > 0$ when $x \in (0, \pi))$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$

$= \frac{\pi}{2} - x/2$

$\therefore y = \frac{\pi}{2} - x/2$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$ Ans

(vi) $y = \cot^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$

$= \cot^{-1}\left(\dfrac{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x}{\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x}\right)$

$= \cot^{-1}\left(\dfrac{\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x}{\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x}\right)$

$= \cot^{-1}\left(\dfrac{\cos\left(\frac{\pi}{4}+x\right)}{\sin\left(\frac{\pi}{4}+x\right)}\right)$

$= \cot^{-1}\left(\cot\left(\frac{\pi}{4}+x\right)\right)$

$= \frac{\pi}{4} + x$

$\therefore y = \frac{\pi}{4} + x$

$\Rightarrow \dfrac{dy}{dx} = 1.$

(vii) $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

$\Rightarrow \sin y = \frac{a+b\cos x}{b+a\cos x}$

diff^n both side w.r.t 'x'

$$\cos y \times \frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$= \frac{(b+a\cos x)(-b\sin x) - (a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$= \frac{-b^2\sin x - \cancel{ab\cos x\sin x} + a^2\sin x + \cancel{ab\sin x\cos x}}{(b+a\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} \times \frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}$$

$$= \frac{\sqrt{b^2+a^2\cos^2 x - a^2 - b^2\cos^2 x}}{(b+a\cos x)}$$

$$= \frac{\sqrt{b^2\sin^2 x - a^2\sin^2 x}}{b+a\cos x}$$

$$= \frac{\sqrt{b^2-a^2}\,\sin x}{b+a\cos x}$$

$$\therefore \frac{dy}{dx} = \frac{b+a\cos x}{\sqrt{b^2-a^2}\,\cancel{\sin x}} \times \frac{(a^2-b^2)\,\cancel{\sin x}}{(b+a\cos x)^2}$$

$$= \frac{-(b^2-a^2)}{\sqrt{b^2-a^2}\,(b+a\cos x)}$$

$$= \frac{-\sqrt{b^2-a^2}}{b+a\cos x} \quad \underline{\text{Ans}}$$

(viii) $y = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan x/2\right)$

$\Rightarrow \tan y = \lambda \tan x/2, \quad \lambda = \sqrt{\frac{a-b}{a+b}}$

$\Rightarrow \sec^2 y \frac{dy}{dx} = \lambda \sec^2 x/2 \times \frac{1}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{\lambda}{2} \cdot \frac{\sec^2 x/2}{\sec^2 y}$

$$= \frac{\lambda}{2} \cdot \frac{\sec^2 x/2}{1+\tan^2 y}$$

$$= \frac{\lambda}{2} \cdot \frac{\sec^2 x/2}{1+\lambda^2 \tan^2 x/2}$$

$$= \frac{1}{2}\sqrt{\frac{a-b}{a+b}} \cdot \frac{\sec^2 x/2}{1+\frac{(a-b)}{(a+b)} \cdot \tan^2 x/2}$$

$$= \frac{1}{2}\sqrt{\frac{a-b}{a+b}} \cdot \frac{(a+b) \cdot \sec^2 x/2}{a+b+a\tan^2 x/2 - b\tan^2 x/2}$$

$$= \frac{1}{2}\sqrt{a^2-b^2} \cdot \frac{1}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2 x/2}$$

$$= \frac{1}{2}\frac{\sqrt{a^2-b^2}}{(a+b\cos x)}$$ Ans

Find $\frac{dy}{dx}$ for the following functions

(i) $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$

(ii) $y = \tan^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

(iii) $y = \tan^{-1}\left(\dfrac{a + b\cos x}{b - a\cos x}\right)$

(iv) $y = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right\}, -1 \le x \le 1$

(v) $y = \cos^{-1}\left(\dfrac{a + b\cos x}{b + a\cos x}\right)$

(vi) $y = \tan^{-1}\left\{\dfrac{\sqrt{a^2-b^2}\,\sin x}{b + a\cos x}\right\}$

(vii) $y = \tan^{-1}\left(\dfrac{4x}{1+5x^2}\right) + \tan^{-1}\left(\dfrac{2+3x}{3-2x}\right)$

(viii) $y = \tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$

Solution

(i) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2} - 1}{x}\right)$

Put $x = \tan\theta$

$\therefore\ y = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$= \tan^{-1}\left(\tan\theta/2\right)$$

$$= \theta/2$$

$$\therefore\ y = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

(ii) $$y = \tan^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$$

$$\Rightarrow \tan y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$$

$$\Rightarrow (1+\sqrt{x})\tan y = 1-\sqrt{x}$$

differentiating both side w.r.t 'x'.

$$\Rightarrow (1+\sqrt{x}).\sec^2 y \frac{dy}{dx} + \tan y . \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}.$$

$$\Rightarrow (1+\sqrt{x})\sec^2 y \frac{dy}{dx} = -(1+\tan y)\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1+\tan y)}{2\sqrt{x}(1+\sqrt{x})\sec^2 y}$$

$$= -\frac{1+\frac{1-\sqrt{x}}{1+\sqrt{x}}}{2\sqrt{x}(1+\sqrt{x})(1+\tan^2 y)}$$

$$= -\frac{\cancel{2}}{\cancel{2}\sqrt{x}(1+\sqrt{x})^2\left(1+\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2\right)}$$

$$= \frac{-1}{\sqrt{x}\,\cancel{(1+\sqrt{x})^2}\,\frac{\{(1+\sqrt{x})^2+(1-\sqrt{x})^2\}}{\cancel{(1+\sqrt{x})^2}}}$$

$$= \frac{-1}{2\sqrt{x}(1+x)}$$ Ans

(iii) $y = \tan^{-1}\left(\dfrac{a+b\cos x}{b-a\cos x}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(\frac{a+b\cos x}{b-a\cos x}\right)^2} \times \frac{d}{dx}\left(\frac{a+b\cos x}{b-a\cos x}\right)$$

$$= \frac{\cancel{(b-a\cos x)^2}}{b^2+a^2\cos^2 x+a^2+b^2\cos^2 x} \times \frac{-(b-a\cos x)b\sin x - (a+b\cos x)a\sin x}{\cancel{(b-a\cos x)^2}}$$

$$= \frac{-b^2\sin x + \cancel{ab\sin x\cos x} - a^2\sin x - \cancel{ab\sin x\cos x}}{(a^2+b^2)(1+\cos^2 x)}$$

$$= \frac{-\cancel{(a^2+b^2)}\sin x}{\cancel{(a^2+b^2)}(1+\cos^2 x)}$$

$$= \frac{-\sin x}{1+\cos^2 x}$$ Ans

(iv) $y = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$

$$= \tan^{-1}\left\{\frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)^2}{1+x^2-(1-x^2)}\right\}$$

$$= \tan^{-1}\left\{\frac{2-2\sqrt{1-x^4}}{2x^2}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\sqrt{1-x^4}}{x^2}\right\}$$

Put $x^2 = \sin\theta$

$$\therefore y = \tan^{-1}\left\{\frac{1-\sqrt{1-\sin^2\theta}}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{2\sin\theta/2\,\cos\theta/2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\theta/2\,\cos\theta/2}\right\}$$

$$= \tan^{-1}\left(\tan\theta/2\right)$$

$$y = \theta/2 = \frac{1}{2}\sin^{-1}x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cancel{2}}\frac{1}{\sqrt{1-x^4}} \times \cancel{2}x$$

$$= \frac{x}{\sqrt{1-x^4}} \quad \underline{\text{Ans}}$$

(v) $$y = \cos^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

<u>Same as (iii)</u>

(vi) $y = \tan^{-1}\left\{\dfrac{\sqrt{a^2-b^2}\,\sin x}{b+a\cos x}\right\}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(\dfrac{\lambda \sin x}{b+a\cos x}\right)^2} \times \frac{d}{dx}\left(\frac{\lambda \sin x}{b+a\cos x}\right)$$

$$(\lambda = \sqrt{a^2-b^2})$$

$$= \frac{\cancel{(b+a\cos x)^2}\,\lambda}{(b+a\cos x)^2+\lambda^2\sin^2 x} \times \frac{(b+a\cos x)\cos x + a\sin x \times \sin x}{\cancel{(b+a\cos x)^2}}$$

$$= \frac{\lambda(b\cos x + a\cos^2 x + a\sin^2 x)}{b^2+a^2\cos^2 x+2ab\cos x+a^2\sin^2 x-b^2\sin^2 x}$$

$$= \frac{\lambda(b\cos x + a)}{b^2\cos^2 x+a^2+2ab\cos x}$$

$$= \frac{\lambda(a+b\cos x)}{(a+b\cos x)^2}$$

$$= \frac{\lambda}{a+b\cos x}$$

$$= \frac{\sqrt{a^2-b^2}}{a+b\cos x}$$ Ans

(vii) $y = \tan^{-1}\left(\frac{4x}{1+5x^2}\right) + \tan^{-1}\left(\frac{2+3x}{3-2x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{4x}{1+5x^2}\right)^2} \times \frac{d}{dx}\left(\frac{4x}{1+5x^2}\right) +$$

$$\frac{1}{1+\left(\frac{2+3x}{3-2x}\right)^2} \times \frac{d}{dx}\left(\frac{2+3x}{3-2x}\right)$$

$$= \frac{\cancel{(1+5x^2)^2} \times 4}{(1+5x^2)^2+16x^2} \times \frac{(1+5x^2 - x \times 10x)}{\cancel{(1+5x^2)^2}} +$$

$$\frac{\cancel{(3-2x)^2}}{(3-2x)^2+(2+3x)^2} \times \frac{(3-2x)\times 3+(2+3x)\,2}{\cancel{(3-2x)^2}}$$

$$= \frac{4(1-5x^2)}{(1+5x^2)^2+16x^2} + \frac{13}{13+13x^2}$$

$$= \frac{4-20x^2}{1+26x^2+25x^4} + \frac{1}{1+x^2}$$

$$= \frac{4-20x^2}{(x^2+1)(25x^2+1)} + \frac{1}{1+x^2}$$

$$= \frac{1}{(1+x^2)}\left\{\frac{4-20x^2}{25x^2+1} + 1\right\}$$

$$= \frac{1}{(1+x^2)} \quad \frac{4-20x^2+25x^2+1}{25x^2+1} = \frac{5\cancel{(1+x^2)}}{\cancel{(1+x^2)}(25x^2+1)}$$

$$= 5 / 25x^2+1 \quad \text{Ans}$$

(viii) $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Put $x = \cos\theta$

$\Rightarrow y = \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$= \tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}$

$= \tan^{-1}(\tan\theta/2)$

$= \theta/2$

$y = \frac{1}{2}\cos^{-1}x.$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{-1}{\sqrt{1-x^2}}$ Ans

Find the derivatives of the following functions

(i) $y = \sin^{-1}(3x-4x^3), \ -\frac{1}{2} < x < \frac{1}{2}$

(ii) $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a(a^2-3x^2)}\right)$

(iii) $y = \cos^{-1}(4x^3-3x), \ -1 < x < -\frac{1}{2}$

(iv) $y = \cos(2\sin^{-1}(\cos x))$

(v) $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right), \; 0<x<1$

(vi) $y = \cos^{-1}\left(8x^2 - 8x^4 - 1\right)$

(vii) $y = \cos^{-1}\left(\dfrac{x^{2n}-1}{x^{2n}+1}\right)$

(viii) $y = \sin\left(\cos^{-1}x\right) + \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Solution

(i) $y = \sin^{-1}\left(3x - 4x^3\right)$

Put $x = \sin\theta$

$\Rightarrow y = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta = 3\sin^{-1}x$

$\therefore \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$ Ans

(ii) $y = \tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

Put $x = a\tan\theta$

$$\therefore y = \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$y = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a}$$

$$= \frac{3a^2}{a^2+x^2} \times \frac{1}{a}$$

$$= \frac{3a}{x^2+a^2} \quad \text{Ans}$$

(iii) $y = \cos^{-1}(4x^3 - 3x)$

Hint: Put $x = \cos\theta$.

$4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} \quad \text{Ans}$$

(iv) $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$

$= \sin^{-1}x + \sin^{-1}\sqrt{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x}} \times \dfrac{d}{dx}\sqrt{x}$

$= \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{2\sqrt{x}\sqrt{1-x}}$ Ans

(v) $y = \cos\left(2\sin^{-1}(\cos x)\right)$

$= \cos\left(2\sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right)\right)$

$= \cos\left(\pi - 2x\right)$

$= -\cos 2x$

$\therefore \dfrac{dy}{dx} = +2\sin 2x = 2\sin 2x$ Ans

(vi) $y = \cos^{-1}(8x^2 - 8x^4 - 1)$

Put $x = \cos\theta$

$\therefore y = \cos^{-1}(8x^2(1-x^2) - 1)$

$= \cos^{-1}(8\cos^2\theta \sin^2\theta - 1)$

$= \cos^{-1}(2.(2\sin\theta\cos\theta)^2 - 1)$

$= -\cos^{-1}(1 - 2\sin^2 2\theta)$

$= -\cos^{-1}(\cos 4\theta)$

$= -4\theta = -4\cos^{-1}x$

$\therefore \frac{dy}{dx} = \frac{4}{\sqrt{1-x^2}}$ Ans

(vii) $y = \cos^{-1}\left(\frac{x^{2n}-1}{x^{2n}+1}\right)$

$\Rightarrow \cos y = \frac{x^{2n}-1}{x^{2n}+1}$

$$\Rightarrow -\sin y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^{2n}-1}{x^{2n}+1}\right)$$

$$= \frac{(x^{2n}+1)\,2nx^{2n-1} - (x^{2n}-1)\,2nx^{2n-1}}{(x^{2n}+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} \times \frac{2nx^{2n-1}(\cancel{x^{2n}}+1-\cancel{x^{2n}}+1)}{(x^{2n}+1)^2}$$

$$= \frac{-4nx^{2n-1}}{\sin y\,(x^{2n}+1)^2}$$

$$= \frac{-4nx^{2n-1}}{(x^{2n}+1)^2\sqrt{1-\left(\frac{x^{2n}-1}{x^{2n}+1}\right)^2}}$$

$$= \frac{-4(x^{2n}+1)\,nx^{2n-1}}{(x^{2n}+1)^2\sqrt{(x^{2n}+1)^2-(x^{2n}-1)^2}}$$

$$= \frac{-4nx^{2n-1}}{(x^{2n}+1)\sqrt{4\cdot x^{2n}\cdot 1}}$$

$$= \frac{-4nx^{2n-1}}{2x^n(x^{2n}+1)} = \frac{-2nx^{2n-1}}{x^n(x^{2n}+1)}$$

$$= \frac{-2nx^{n-1}}{(1+x^{2n})}$$ Ans

(viii) $y = \sin(\cos^{-1}x) + \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= \sin\left(\sin^{-1}\sqrt{1-x^2}\right) + \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= \sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\therefore \frac{dy}{dx} = \frac{(-2x)}{2\sqrt{1-x^2}} + \frac{1}{2}\frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)$

$= \frac{-x}{\sqrt{1-x^2}} + \frac{1}{2}\frac{d}{d\theta}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\frac{d\theta}{dx}$

Put $x = \tan\theta$

$= \frac{-x}{\sqrt{1-x^2}} + \frac{1}{2}\frac{d}{d\theta}2\theta \times \frac{1}{\sec^2\theta}$

$= \frac{-x}{\sqrt{1-x^2}} + \frac{1}{1+\tan^2\theta}$

$= \frac{-x}{\sqrt{1-x^2}} + \frac{1}{1+x^2}$ Ans