Complex Numbers and Quadratic Equations

1. If $a=\cos\alpha+i\sin\alpha, b=\cos\beta+i\sin\beta, c=\cos\gamma+i\sin\gamma$ and $\frac{b}{c}+\frac{c}{a}+\frac{a}{b}=1$, prove that $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=1$

Solution:

$$\frac{b}{c}+\frac{c}{a}+\frac{a}{b}=1$$

$$\Rightarrow bc^{-1}+ca^{-1}+ab^{-1}=1$$

$$\Rightarrow(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)^{-1}+(\cos\gamma+i\sin\gamma)(\cos\alpha+i\sin\alpha)^{-1}$$

$$+(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)^{-1}=1$$

$$\Rightarrow(\cos\beta+i\sin\beta)(\cos(-\gamma)+i\sin(-\gamma))+(\cos\gamma+i\sin\gamma)(\cos(-\alpha)+i\sin(-\alpha))$$

$$+(\cos\alpha+i\sin\alpha)(\cos(-\beta)+i\sin(-\beta))=1$$

$$\Rightarrow\cos(\beta-\gamma)+i\,sin(\beta-\gamma)+\cos(\gamma-\alpha)+i\,sin(\gamma-\alpha)$$

$$+\cos(\alpha-\beta)+i\,sin(\alpha-\beta)=1+i0$$

$$\Rightarrow\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=1$$

(Equating real part on both side)

2. Find the value of $\sqrt{-3+\sqrt{-3+\sqrt{-3+.....to\inf inity}}}$

Solution: Let $z=\sqrt{-3+\sqrt{-3+\sqrt{-3+.....to\inf inity}}}$

$$\Rightarrow z=\sqrt{-3+z}$$

$$\Rightarrow z^2=-3+z$$

$$\Rightarrow z^2-z+3=0$$

$$\Rightarrow z=\frac{1\pm\sqrt{1-12}}{2}$$

$$\Rightarrow z=\frac{1\pm i\sqrt{11}}{2}$$

3. Show that $1+\omega+\omega^2+.....+\omega^{n-1}=1+i\cot\left(\frac{\pi}{2n}\right)$, where $\omega=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$

Solution:

$$1+\omega+\omega^2+.....+\omega^{n-1}=\frac{1-\omega^n}{1-\omega}$$

$$=\frac{1-\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)^n}{1-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}}=\frac{1-\cos\pi-i\sin\pi}{2\sin^2\frac{\pi}{2n}-2i\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}$$

$$=\frac{2}{-2i\sin\frac{\pi}{2n}\left(\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}\right)}$$

$$=\frac{-1}{i\sin\frac{\pi}{2n}\left(\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}\right)}=\frac{-\left(\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}\right)^{-1}}{i\sin\frac{\pi}{2n}}$$

$$=\frac{-\left(\cos\frac{\pi}{2n}-i\sin\frac{\pi}{2n}\right)}{i\sin\frac{\pi}{2n}}=-\frac{1}{i}\left(\cot\frac{\pi}{2n}-i\right)=1+i\cot\left(\frac{\pi}{2n}\right)$$

4. Find the amplitude of the complex number -3-3i

Solution: $amp(-3-3i)=\tan^{-}\frac{-3}{-3}-\pi=\frac{\pi}{4}-\pi=-\frac{3\pi}{4}$

Remember:

$$amp(z)=\begin{cases}\tan^{-1}\frac{y}{x}, x>0\\ \tan^{-1}\frac{y}{x}+\pi, x<0, y\geq 0\\ \tan^{-1}\frac{y}{x}-\pi, x<0, y<0\\ \frac{\pi}{2}, x=0, y>0\\ -\frac{\pi}{2}, x=0, y<0\end{cases}$$

5. Show that the number of distinct elements in the set $S=\{i^n+i^{-n}:n\in Z\}$ is 3

Solution: By putting n by $-n$ we see that $i^n+i^{-n}=i^{-n}+i^{-(-n)}=i^{-n}+i^n$. So it is sufficient to vary n over natural numbers together with 0.

For n=0 $i^n+i^{-n}=2$

If n is even and greater than 0

$i^n+i^{-n}=i^{2k}+i^{-2k}, k\in N$

$=(-1)^k+(-1)^{-k}, k\in N$

$=2, if\ k\ is\ even$

$=-2, if\ k\ is\ odd$

If n is odd

$i^n + i^{-n} = i^{2k+1} + i^{-2k-1}, k \in N$

$= (-1)^k i + (-1)^{-k} i^{-1}, k \in N$

$= (-1)^k i - i(-1)^k$

$= 0, if\ k\ is\ even$

$= 0, if\ k\ is\ odd$

So there are three distinct values namely 0,2,-2

6. If $\tan A$ and $\tan B$ are the roots of the equation $x^2 - px + q = 0$, show that

$$\sin^2(A+B) = \frac{p^2}{(1-q)^2 + p^2}$$

Solution:

$\tan A + \tan B = p$

$\tan A \tan B = q$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{p}{1-q}$$

now

$$\sin^2(A+B) = \frac{\tan^2(A+B)}{\sec^2(A+B)} = \frac{\frac{p^2}{(1-q)^2}}{1 + \frac{p^2}{(1-q)^2}}$$

$$= \frac{p^2}{(1-q)^2 + p^2}$$

7. If a,b,c are real numbers and (a+b+c)c<0, then show that $ax^2 + bx + c = 0$ has two real roots.

Solution:

$(a+b+c)c < 0$

$\Rightarrow ac + bc + c^2 < 0$

$\Rightarrow ac < -bc - c^2$

$\Rightarrow -4ac > 4bc + 4c^2$

$\Rightarrow b^2 - 4ac > b^2 + 4bc + 4c^2$

$\Rightarrow b^2 - 4ac > (b+2c)^2 \geq 0$

therefore roots are real

8. If one of the roots of the equation $x^2 + px + q = 0$ is square of the other show that $p^3 - q(3p-1) + q^2 = 0$

Solution: we have

$\alpha + \alpha^2 = -p$

$\alpha^3 = q$

We have to eliminate α between the above two equation

$$-p^3 = \alpha^3(\alpha+1)^3$$
$$\Rightarrow -p^3 = q(\alpha^3 + 1 + 3\alpha^2 + 3\alpha)$$
$$\Rightarrow -p^3 = q(q + 1 - 3p)$$

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